Relationships between the old Gauss-Krüger projection and UTM projection for Croatia

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Abstract
The paper gives a complete set of formulas for transformation from the old Gauss-Krüger map projection of Croatia to the UTM and vice versa. One needs to have these transformations at disposal for the print preparation of the UTM plane grid on the sheets of map produced in the old Gauss-Krüger projection.
Keywords: map projections, Gauss-Krüger, UTM, military maps, NATO

Introduction

In Croatia there are preparations made for the transfer from the old official geodetic datum and map projections into the new ones. The inherited map projection is Gauss-Krüger of 3 degrees meridian zones with linear scale 0.9999 along the central meridian. Universal Transverse Mercator – UTM is similar to Gauss-Krüger projection by being conformal, transverse and cylindrical, and it differs from it by being the projection of 6 deg. meridian zones with linear scale 0.9996 along the central meridian.

A large number of states, as well as the alliance of states (NATO) have accepted UTM as their official projection. In this paper there are clear relationships given between the old Gauss-Krüger and UTM projections for the Croatian territory.

The three sheets of Croatian military topographic map at the scale of 1:50 000 were on display at the EUREF 2001 meeting in Dubrovnik, and were chosen to illustrate the position of new UTM grid printed over the old contents.

Basic characteristics of military maps of the NATO members

The main directions in standardization in the filed of map production in NATO member states are collected in the appropriate Standardization Agreements (STANAG). In creating a uniform designed maps in Western and Central Europe the following have been accomplished:

- the unique European geodetic system ED 50 (European Datum 1950) with the International Hayford ellipsoid as a reference surface or the World Geodetic System (WGS 84) with the ellipsoid WGS 84 as a reference surface;
- the uniform map presentation reached by using the Universal Transverse Mercator (UTM) map projection of 6 degrees meridian zones;
- UTM coordinate system for determining cartesian plane coordinates of points and objects by using the uniform military grid.

The standardization of maps of NATO member states is made in such a way that the geodetic coordinates of corners of the official topographic sheets at the scale 1:50 000 (very rarely 1:25 000 and 1:100 000) are transformed in the recommended geodetic system (ED 50 or

WGS 84 for the continental part of Europe). Moreover, the cartesian plane UTM grid is additionally printed on these sheets.

The use of geodetic coordinate systems in the NATO framework is not completely uniform. More detail information can be found in *Geodetic datums, ellipsoids, grids and grid references*, STANAG 2211.

Any NATO member state retains its individual map graphics, and the contents, accuracy and the sheets numbering remained without changes as well. In contrast to civil official maps, the data on the sheet margins are enlarged first of all by hints to the determination of plane coordinates in the military UTM grid, then by the review of main symbols, used abbreviations and some additional information. The text in the sheet margins is written in a national language, as well as in English.

Transformations between the Gauss-Krüger and UTM projections for Croatia

In order to print the UTM plane grid on the sheet of map produced in Gauss-Krüger map projection for the territory of Croatia, one needs to have the following transformations at disposal:

$\boxed{1}$ From the coordinates X, Y in the plane of Gauss-Krüger projection to compute the appropriate geodetic coordinates φ, λ of the Bessel ellipsoid.

1 From the coordinates E, N in the plane of UTM projection to compute the appropriate geodetic coordinates φ, λ of the WGS 84 ellipsoid.

$\boxed{2}$ From the geodetic coordinates φ, λ of the Bessel ellipsoid to compute the spatial cartesian coordinates x_{Bessel}, y_{Bessel}, z_{Bessel}.

2 From the geodetic coordinates φ, λ of the WGS 84 ellipsoid to compute the spatial cartesian coordinates x_{WGS84}, y_{WGS84}, z_{WGS84}.

$\boxed{3}$ From the spatial cartesian coordinates x_{Bessel}, y_{Bessel}, z_{Bessel} to compute the spatial cartesian coordinates x_{WGS84}, y_{WGS84}, z_{WGS84}. The needed seven parameters one can take over from (Bilajbegović and Podunavac, 1993).

3 From the spatial cartesian coordinates x_{WGS84}, y_{WGS84}, z_{WGS84} to compute the spatial cartesian coordinates x_{Bessel}, y_{Bessel}, z_{Bessel}. The needed seven parameters one can take over from (Bilajbegović and Podunavac, 1993).

$\boxed{4}$ From the spatial cartesian coordinates x_{WGS84}, y_{WGS84}, z_{WGS84} to compute the geodetic coordinates φ, λ of the WGS 84 ellipsoid. It can be done by formulas based on (Lapaine, 1991).

4 From the spatial cartesian coordinates x_{Bessel}, y_{Bessel}, z_{Bessel} to compute the geodetic coordinates φ, λ of the Bessel ellipsoid. It can be done by formulas based on (Lapaine, 1991).

$\boxed{5}$ From the geodetic coordinates φ, λ of the WGS84 ellipsoid to compute the coordinates E, N in the plane of UTM projection.

5 From the geodetic coordinates φ, λ of the Bessel ellipsoid to compute the coordinates X, Y in the plane of Gauss-Krüger projection.

All formulas needed for the practical implementation of the mentioned transformations could be the following:

1 **①**

$a = 6\ 377\ 397.155$
$b = 6\ 356\ 078.96325$
$m_0 = 0.9999$
<u>5:</u> $Y_0 = 5\ 000\ 000$, $\lambda_0 = 15°$
<u>6:</u> $Y_0 = 6\ 000\ 000$, $\lambda_0 = 18°$

$$Y' = \frac{(Y - Y_0 - 500\ 000)}{m_0},\ X' = \frac{X}{m_0}$$

$a = 6\ 378\ 137$
$b = 6\ 356\ 752.31425$
$m_0 = 0.9996$
<u>33T:</u> $\lambda_0 = 15°$
<u>34T:</u> $\lambda_0 = 21°$

$$Y' = \frac{(E - 500\ 000)}{m_0},\ X' = \frac{N}{m_0}$$

$$e = \sqrt{1 - \frac{b^2}{a^2}},\ e' = \sqrt{\frac{a^2}{b^2} - 1}$$

$$\mu_1 = \frac{X'}{a(1 - e^2/4 - 3e^4/64 - 5e^6/256)},\ e_1 = \frac{1-(1-e^2)^{1/2}}{1+(1-e^2)^{1/2}}$$

$$\varphi_1 = \mu_1 + \left(\frac{3}{2}e_1 - \frac{27}{32}e_1^3\right)\sin 2\mu_1 + \left(\frac{21}{16}e_1^2 - \frac{55}{32}e_1^4\right)\sin 4\mu_1 + \frac{151}{96}e_1^3 \sin 6\mu_1 + \frac{1097}{512}e_1^4 \sin 8\mu_1$$

$$N_1 = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_1}},\ M_1 = \frac{a\left(1-e^2\right)}{\left(1 - e^2\sin^2\varphi_1\right)^{3/2}},\ T_1 = \tan^2\varphi_1,\ C_1 = e'^2\cos^2\varphi_1,\ D = \frac{Y'}{N_1}$$

$$\varphi = \varphi_1 - \frac{N_1 \tan\varphi_1}{M_1}\left[\begin{array}{l}\frac{D^2}{2} - \left(5 + 3T_1 + 10C_1 - 4C_1^2 - 9e'^2\right)\frac{D^4}{24} \\ + \left(61 + 90T_1 + 298C_1 + 45T_1^2 - 252e'^2 - 3C_1^2\right)\frac{D^6}{720}\end{array}\right]$$

$$\lambda = \lambda_0 + \frac{1}{\cos\varphi_1}\left[D - \left(1 + 2T_1 + C_1\right)\frac{D^3}{6} + \left(5 - 2C_1 + 28T_1 - 3C_1^2 + 8e'^2 + 24T_1^2\right)\frac{D^5}{120}\right]$$

φ_{Bessel}, λ_{Bessel} φ_{WGS84}, λ_{WGS84}

2 **②**

$a = 6\ 377\ 397.155$
$b = 6\ 356\ 078.96325$
$\varphi = \varphi_{Bessel}$, $\lambda = \lambda_{Bessel}$

$a = 6\ 378\ 137$
$b = 6\ 356\ 752.31425$
$\varphi = \varphi_{WGS84}$, $\lambda = \lambda_{WGS84}$

$$N = \frac{a}{\sqrt{1 - e^2\sin^2\varphi}},\ x = N\cos\varphi\cos\lambda,\ y = N\cos\varphi\sin\lambda,\ z = \frac{N}{1-e^2}\sin\varphi$$

x_{Bessel}, y_{Bessel}, z_{Bessel} x_{WGS84}, y_{WGS84}, z_{WGS84}

$\boxed{3}$

$$\vec{r}_{Bessel} = \begin{bmatrix} x_{Bessel} & y_{Bessel} & z_{Bessel} \end{bmatrix}$$

3

$$\vec{r}_{WGS84} = \begin{bmatrix} x_{WGS84} & y_{WGS84} & z_{WGS84} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\beta\cos\gamma & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix}$$

$$\vec{r}_{WGS84} = (1 - dm)\cdot R^T \cdot (\vec{r}_{Bessel} - \vec{a})$$

$$\vec{r}_{Bessel} = \vec{a} + (1 + dm)\cdot R \cdot \vec{r}_{WGS84}$$

$\boxed{4}$

$x = x_{WGS84}$, $y = y_{WGS84}$, $z = z_{WGS84}$

4

$x = x_{Bessel}$, $y = y_{Bessel}$, $z = z_{Bessel}$

$$\rho = \sqrt{x^2 + y^2}\,,\ \zeta = |z|\,,\ sgz = \mathrm{sgn}(z)$$

If $\rho < a - \dfrac{b^2}{a}$ and $\zeta = 0$, then φ is not defined.

If $\rho \geq a - \dfrac{b^2}{a}$ and $\zeta = 0$, then $\varphi = 0$ and $h = \rho - a$.

If $\rho = 0$ and $\zeta \neq 0$, then $\varphi = sgz\dfrac{\pi}{2}$ and $h = \zeta - b$.

If $\rho > \zeta$ then $m = \dfrac{a\rho + a^2 - b^2}{b\zeta}$, $n = \dfrac{a\rho + b^2 - a^2}{b\zeta}$, and after (*):

$$\varphi = \arctan\left(sgz\frac{2x}{1-x^2}\frac{a}{b} \right),\ h = \frac{\rho}{\cos\varphi} - \frac{a}{\sqrt{1 - e^2\sin^2\varphi}}.$$

If $\rho \leq \zeta$ then $m = \dfrac{b\zeta + b^2 - a^2}{a\rho}$, $n = \dfrac{b\zeta + a^2 - b^2}{a\rho}$, and after (*):

$$\psi = \arctan\left(sgz\frac{2x}{1-x^2}\frac{b}{a} \right),\ \varphi = \frac{\pi}{2} - \psi\,,\ h = \frac{\zeta}{\cos\psi} - \frac{b}{\sqrt{1 + e'^2\sin^2\psi}}.$$

(*):

$$p = (mn+1)/3\,,\ q = (m^2 - n^2)/4\,,\ D = q^2 + p^3$$

If $D \ge 0$ then $y = \sqrt[3]{\sqrt{D} - q} - \sqrt[3]{\sqrt{D} + q}$.

If $D < 0$ then $r = \operatorname{sgn}(q)\sqrt{|p|}$,

if $r = 0$ then $y = 0$, if $r \ne 0$ then $\alpha = \arccos\left(\frac{q}{r^3}\right)$, $y = -2r\cos\frac{\alpha}{3}$.

$$t = y - \sqrt{y^2 + 1}\ ,\quad s = \frac{n - mt}{2\sqrt{y^2 + 1}}\ ,\quad x = \frac{-t}{s + \sqrt{s^2 - t}}\ .$$

$\varphi_{WGS84}, \lambda_{WGS84}$

$\varphi_{Bessel}, \lambda_{Bessel}$

$\boxed{5}$

5

$\varphi = \varphi_{WGS84}$, $\lambda = \lambda_{WGS84}$
<u>33T:</u> $\lambda_0 = 15°$
<u>34T:</u> $\lambda_0 = 21°$

$\varphi = \varphi_{Bessel}$, $\lambda = \lambda_{Bessel}$
<u>5:</u> $\lambda_0 = 15°$
<u>6:</u> $\lambda_0 = 18°$

$$A = (\lambda - \lambda_0)\cos\varphi\ ,\quad T = \tan^2\varphi\ ,\quad C = e'^2\cos^2\varphi\ ,\quad N = \frac{a}{\sqrt{1 - e^2\sin^2\varphi}}$$

$$M = a\left[\left(1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6\right)\varphi - \left(\frac{3}{8}e^2 + \frac{3}{32}e^4 + \frac{45}{1024}e^6\right)\sin 2\varphi \right.$$
$$\left. + \left(\frac{15}{256}e^4 + \frac{45}{1024}e^6\right)\sin 4\varphi - \frac{35}{3072}e^6\sin 6\varphi\right]$$

$$Y' = N\left[A + (1 - T + C)\frac{A^3}{6} + \left(5 - 18T + T^2 + 72C - 58e'^2\right)\frac{A^5}{120}\right]$$

$$X' = M + N\tan\varphi\left[\frac{A^2}{2} + \left(5 - T + 9C + 4C^2\right)\frac{A^4}{24} + \left(61 - 58T + T^2 + 600C - 330e'^2\right)\frac{A^6}{720}\right]$$

$m_0 = 0.9996$
UTM zone $33T$
UTM zone $34T$
$E = Y'm_0 + 500\ 000$, $N = X'm_0$

$m_0 = 0.9999$
<u>5:</u> $Y_0 = 5\ 000\ 000$
<u>6:</u> $Y_0 = 6\ 000\ 000$
$Y = Y'm_0 + Y_0 + 500\ 000$, $X = X'm_0$

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