Reference Frames and Gravity, in Science and Applications

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EUREF Symposium 2015

Overview

- 1915-2015: one century of General Relativity, a geometric approach to Gravity
- Several ideas behind gravity as geometric curvature are in common with the theory of cartography and leveling
- Motion in a curved spacetime affects massive and massless particles
- As Reference Frames become more globally defined, we must start considering how gravity affects 4D coordinates at the scale of the curvature of the terrestrial gravity field ($Gm/c^2r \sim 0.7$ ppb)

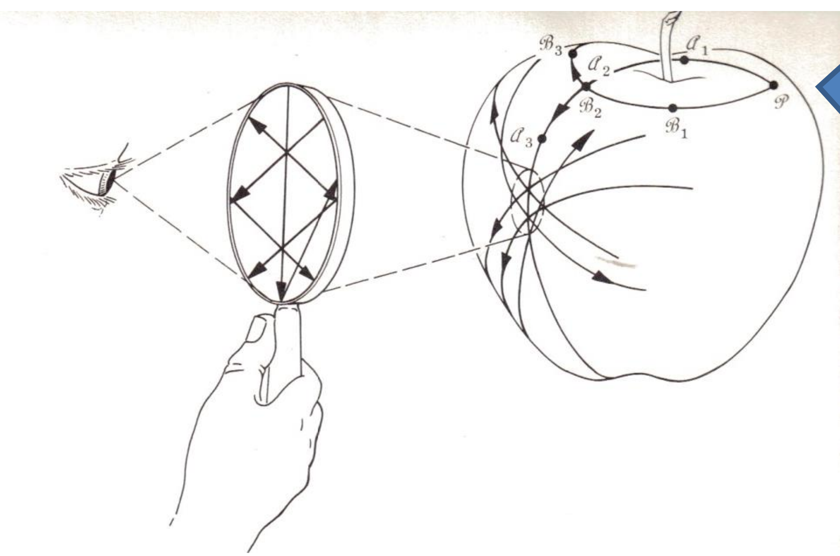
1915-2015: one century of General Relativity (= geometric theory of gravity)

- Einstein, A. (1915): *Zur allgemeinen Relativitaetstheorie.* Preuss. Akad. Wiss. Berlin Sitzber., pp. 778-786
- Einstein, A. (1915): *Zur allgemeinen Relativitaetstheorie (Nachtrag).* Preuss. Akad. Wiss. Berlin Sitzber., pp. 799-801
- Einstein, A. (1915): *Erklaerung des Perihelbewegung des Merkur aus der allgemeinen Relativitaetstheorie.* Preuss. Akad. Wiss. Berlin Sitzber., pp. 831-839
- Einstein, A. (1915): *Die Feldgleichungen der Gravitation.* Preuss. Akad. Wiss. Berlin Sitzber., pp. 844-847

Why is General Relativity so important to Geodesy (..and viceversa)?

'… One day in the year 1666 Newton had gone to the country and seeing the fall of an apple, as his niece told me, let himself be let in a deep meditation on the cause which thus draws every object along a line whose extension would pass almost through the center of the Earth..'

Voltaire, F.M. (1738), *Elements de Philosophie de Newton*, Pt.3, chapter III



Apple as a first example of curved space, which locally looks flat:

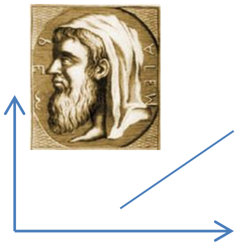
Two curves A and B initially starting from P at a divergent angle, eventually cross and continue in different directions

Locally, the behavior the geometry of the two curves on the curved surface of the apple looks exactly as in Euclidean 'flat' (i.e. not curved) space.

Geodesic as a line of minimum length

Euclides→Riemann→ Newton, Einstein

Euclidean geometry



$$\frac{d\vec{v}}{dt}=0$$

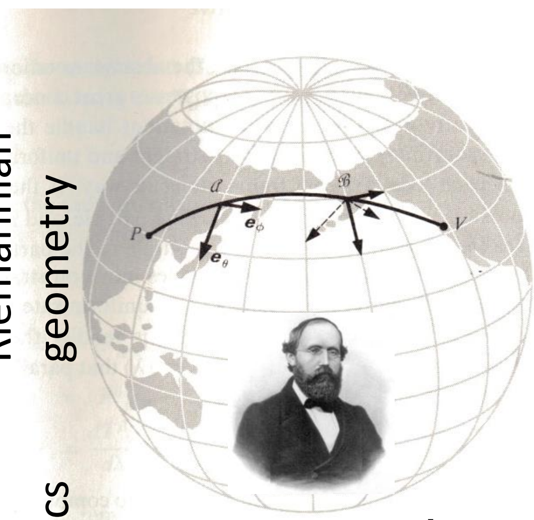
$$ds^2 = dx^2+dy^2$$

$$\int ds = \text{minimum}$$

Geodesic as parallel transport in a manifold:

- in a flat manifold the ordinary derivative of the tangent vector is zero -> trajectory is a straight line. ‘t’ is an affine parameter along the trajectory

Riemannian geometry



$$\frac{D\vec{v}}{Dt}=0$$

$$ds^2 = g_{ij}dx^i dx^j$$

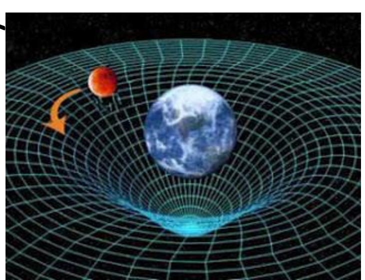
$$\int ds = \text{minimum}$$

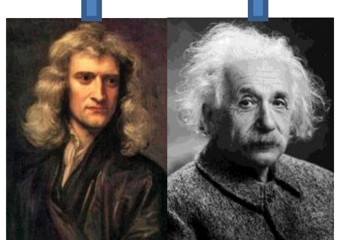
- In a curved manifold the minimum length trajectory (great circle) bends. The covariant derivative of the tangent vector is zero. Curvature is defined by the gravitational and centrifugal potential. ‘t’ is an affine parameter along the trajectory

Christoffel symbols express the force in terms of geometric curvature

$$\frac{dv_i}{dt} = -\frac{\partial U}{\partial x^i} \Leftrightarrow \frac{Dv_i}{D\tau} \equiv \frac{dv_i}{d\tau} + \Gamma^j_{ik} v_j v^k = 0$$

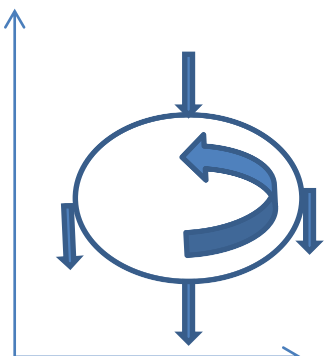
Einstein Geometrodynamics



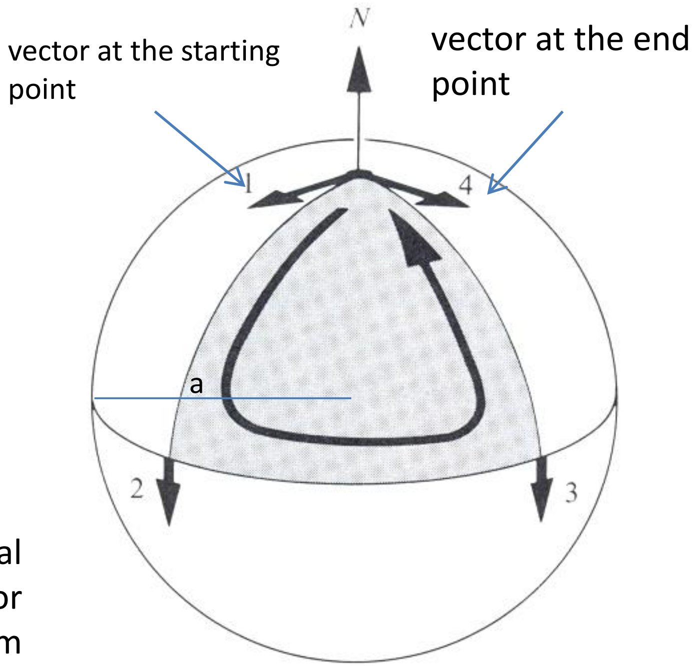


- Dynamics: the Newton Equation of motion and that of a geodesic in curved space time coincide, to a first approximation. The covariant derivative of the tangent vector (= velocity) is zero. Curvature is defined by the gravitational potential. ‘τ’ is an affine parameter along the trajectory and coincides with proper time

How spatial curvature can be measured by parallel transport of a vector along a loop

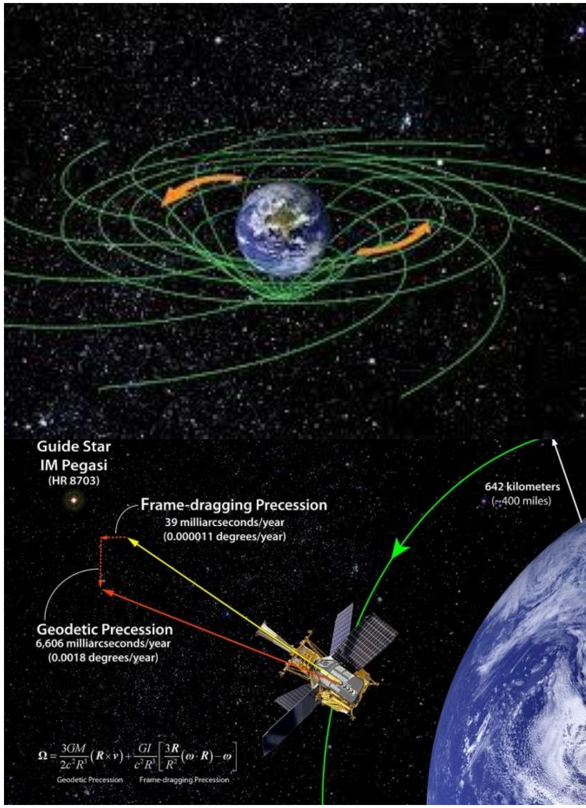


In flat space the initial and final direction of the vector transported by parallelism along a loop coincide



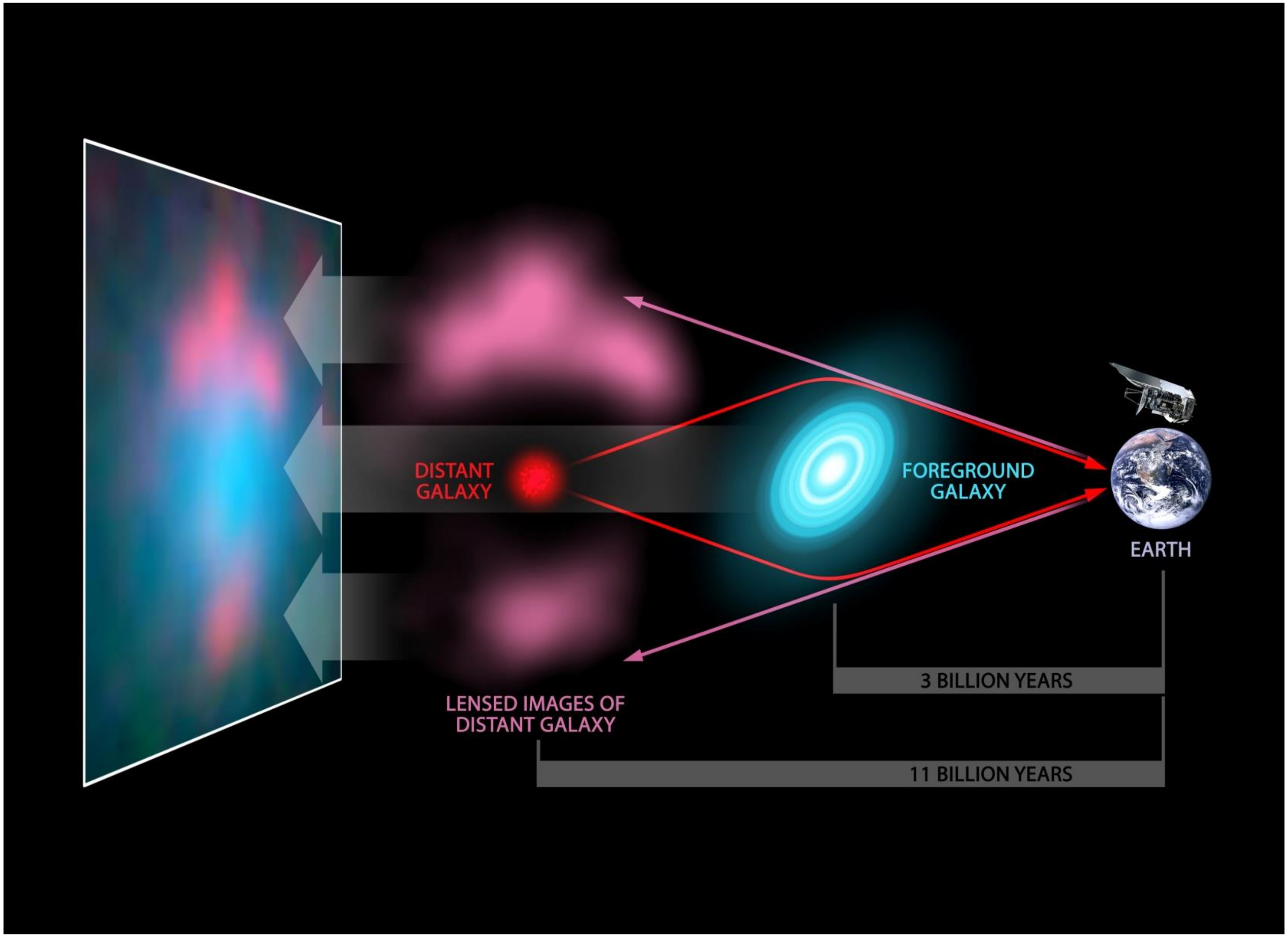
In curved space the rotation angle resulting from parallel translation of a vector along a close circuit is inversely proportional to the square radius of curvature of the manifold.

$$(\text{Gaussian curvature}) = \frac{\text{angle 1 - 4}}{\text{area of path}} = \frac{\frac{\pi}{2}}{\left(\frac{1}{8}\right)\times 4\pi a^2} = \frac{1}{a^2}$$

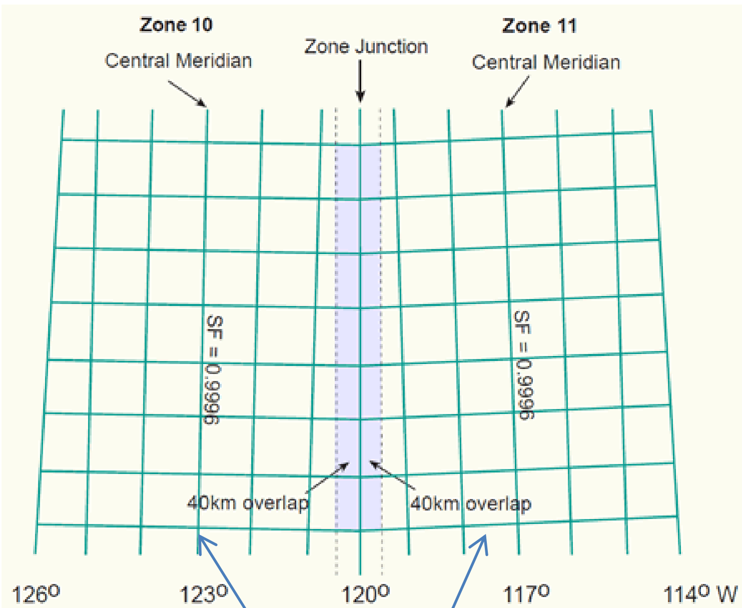


In a curved spacetime with a rotating central body the rotation of a vector parallely transported along a geodesic will be the sum of the geodetic precession and an additional rotation due to ‘Frame Dragging’

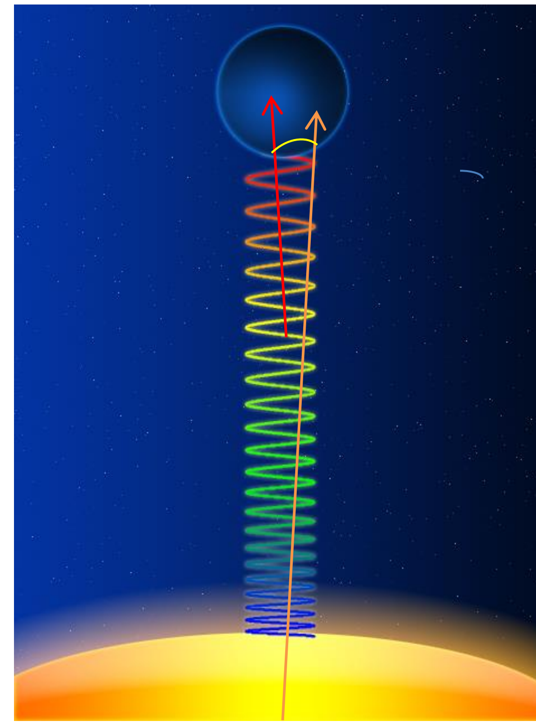
The trajectories of light rays (= massless photon) follow bending geodesics



Coordinate frames can be defined only locally: 2D horizontal, 1D vertical, time

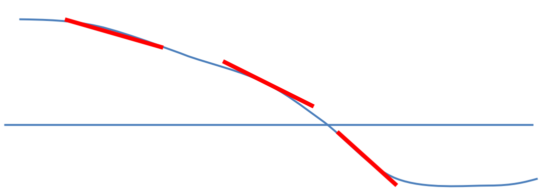


The two grids are not equivalent because tangent at different points (central meridians) of a curved manifold



For GNSS: at each point of the orbit (=accelerated frame) the frequency of the on board clock must be corrected depending on local gravity. Two events A nd B are synchronous with time scale T but happen at a different time when measured with the time scale T', with an acceleration tacking place between T and T'

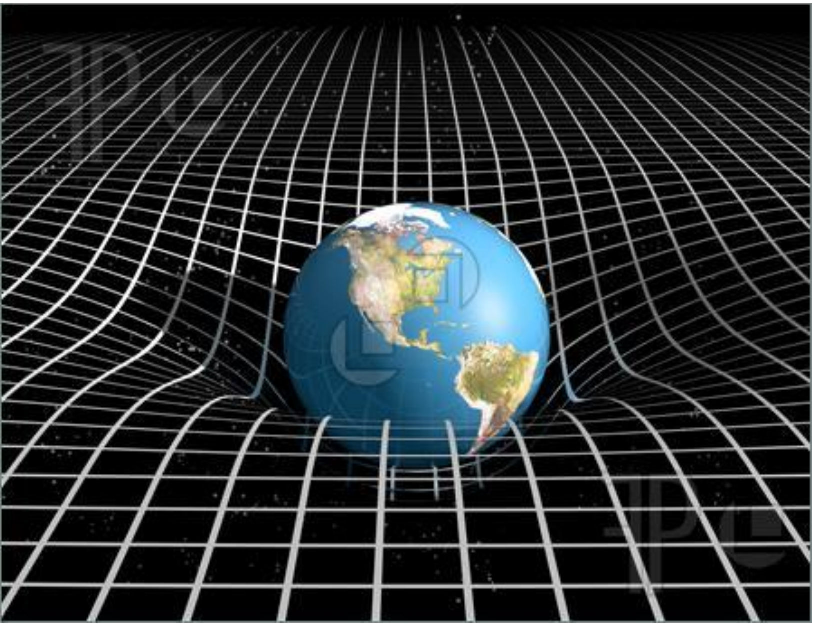
W_0=const

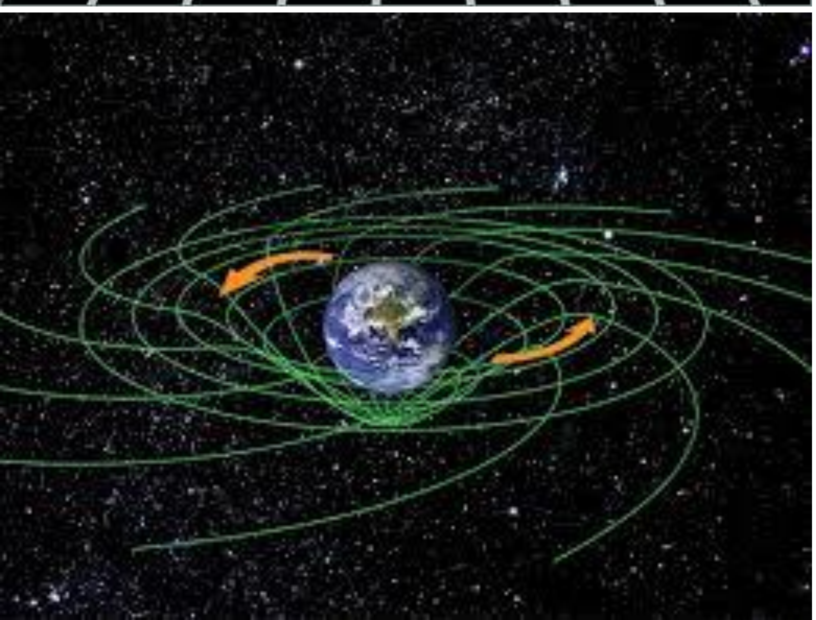


U_0=const

In leveling, the curvature of the equipotential forces tangent planes at different places to be not parallel

Conclusion





Coordinate lines are not rectilinear due to the curvature of spacetime; the unit interval stretches from point to point as a function of gravity

Comparing a coordinate frame with origin at the surface of the earth with a coordinate frame at infinity implies a scale factor of the order of 0.1 ppb, and rotations of a fraction of milliarcsec., due to the curvature of the intervening spacetime

Likewise comparing clocks in regions of different gravity implies gravity dependent corrections

When the rotation of the Central Body is taken into account, curvature of space and time are coupled to each other (Lense Thirring effect).

Prediction:

In the future, we will need ***'gravity related coordinates'*** (and not just heights and timing) , with an indication of the value of the local gravity