Integer Ambiguity Resolution for Precise Point Positioning

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Overview

- Introduction
- Sequential Best-Integer Equivariant Estimation
- Multi-frequency code carrier linear combinations
- Galileo:
  Reliable single epoch integer ambiguity resolution
  with E1/E5 linear combinations

Precise Point Positioning

1. Estimation of fractional widelane bias with Melbourne-Wübbena combination
2. Estimation of ionosphere-free phase clocks and pseudorange clocks with a Kalman filter
3. Broadcast of widelane biases, orbit corr. and ionosphere-free phase/ pseudorange clocks

network of reference stations

4. Estimation of widelane ambiguities from Melbourne-Wübbena combination
5. Computation of ionosphere-free combinations and subtraction of clock parameters
6. Estimation of absolute position, tropospheric delay and ambiguites

mobile receiver

Precise Point Positioning with integer ambiguity resolution

Model of ionosphere-free combination
of satellite-satellite single difference measurements:

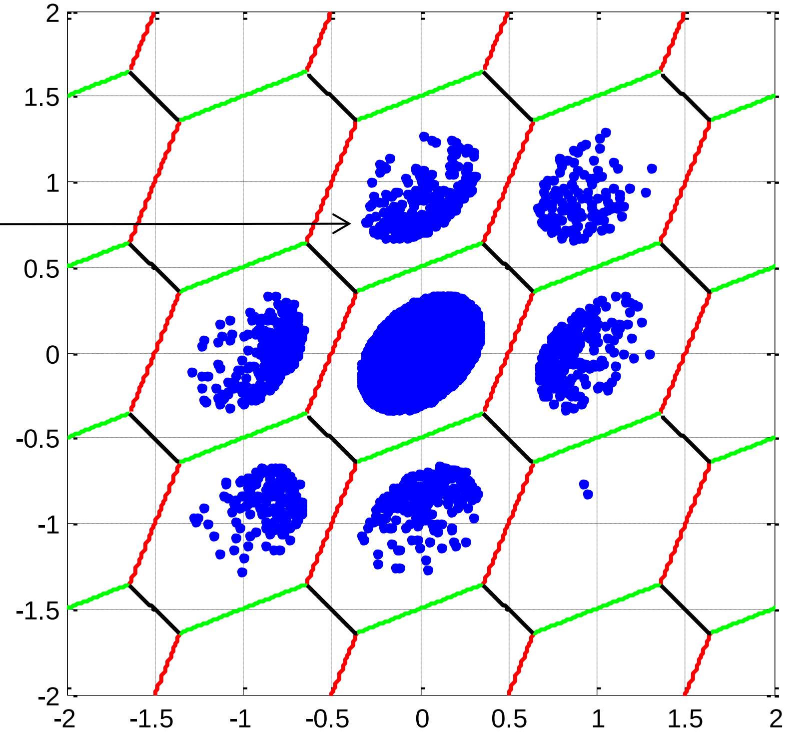
$$\underbrace{\begin{bmatrix} \lambda\tilde{\phi}^{12}_{u,\mathrm{IF}} \\ \lambda\tilde{\phi}^{13}_{u,\mathrm{IF}} \\ \vdots \\ \lambda\tilde{\phi}^{1K}_{u,\mathrm{IF}} \\ \hline \lambda\tilde{\rho}^{12}_{u,\mathrm{IF}} \\ \lambda\tilde{\rho}^{13}_{u,\mathrm{IF}} \\ \vdots \\ \lambda\tilde{\rho}^{1K}_{u,\mathrm{IF}} \end{bmatrix}}_{\boldsymbol{\Psi}} = \underbrace{\left[\begin{array}{c|c|ccc} \Delta\vec{\boldsymbol{e}}^{12}_u & m(\theta^1_u) - m(\theta^2_u) & \lambda_{\mathrm{NL}} & & \\ \vdots & \vdots & & \ddots & \\ \Delta\vec{\boldsymbol{e}}^{1K}_u & m(\theta^1_u) - m(\theta^K_u) & & & \lambda_{\mathrm{NL}} \\ \hline \Delta\vec{\boldsymbol{e}}^{12}_u & m(\theta^1_u) - m(\theta^1_u) & & & \\ \vdots & \vdots & & \mathbf{0} & \\ \Delta\vec{\boldsymbol{e}}^{1K}_u & m(\theta^1_u) - m(\theta^K_u) & & & \end{array}\right]}_{[\boldsymbol{H},\ \boldsymbol{A}]} \underbrace{\begin{bmatrix} \vec{\boldsymbol{x}}_u \\ T_{z,u} \\ N^{12}_{u,1} \\ \vdots \\ N^{1K}_{u,1} \end{bmatrix}}_{\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{N} \end{bmatrix}} + \underbrace{\begin{bmatrix} \varepsilon^{12}_{u,\mathrm{IF}} \\ \varepsilon^{13}_{u,\mathrm{IF}} \\ \vdots \\ \varepsilon^{1K}_{u,\mathrm{IF}} \\ \hline \eta^{12}_{u,\mathrm{IF}} \\ \eta^{13}_{u,\mathrm{IF}} \\ \vdots \\ \eta^{1K}_{u,\mathrm{IF}} \end{bmatrix}}_{\boldsymbol{\eta}}$$

Integer least-squares versus Integer Aperture Estimation

Integer least-squares:

$$S_{\check{N}_k} = \left\{ \hat{\mathbf{N}} \in \mathbb{R}^K \,|\, \check{\mathbf{N}}_k \stackrel{!}{=} \arg\min_{\mathbf{N}} \|\hat{\mathbf{N}} - \mathbf{N}\|^2_{\mathbf{\Sigma}^{-1}_{\hat{N}}} \right\}$$

Pull-in region



Integer aperture estimation:

$$S_{\check{N}_k} = \left\{ \hat{\mathbf{N}} \in \mathbb{R}^K \,|\, \|\hat{\mathbf{N}} - \check{\mathbf{N}}_k\|^2_{\mathbf{\Sigma}^{-1}_{\hat{N}}} \leq \mu^2 \right\}$$

controls probability of wrong fixing

Integer least-squares versus Integer Aperture Estimation

Integer least-squares:

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Integer aperture estimation:

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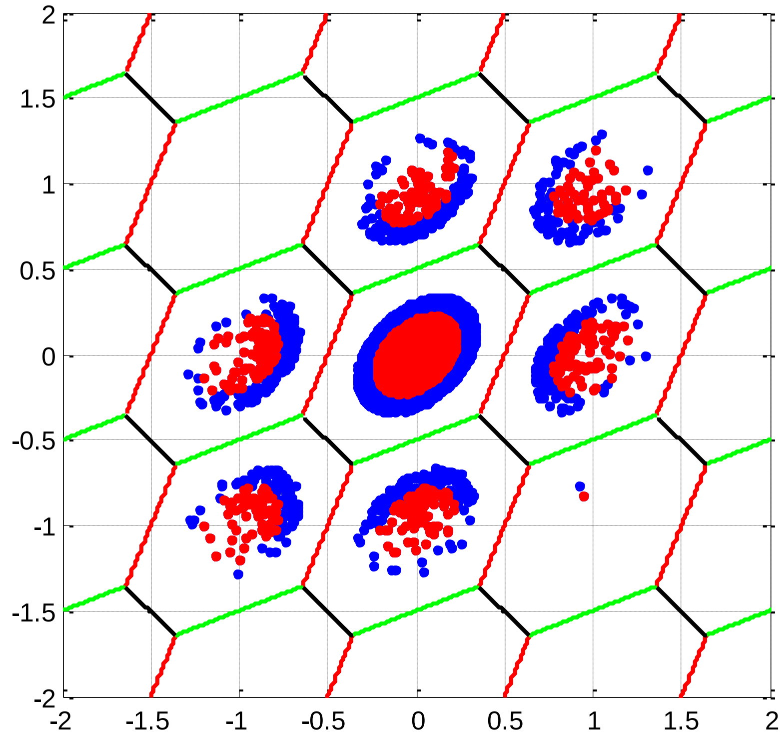

Best Integer Equivariant Estimation

Minimization of Mean Square Error (MSE):

$$\begin{pmatrix} \check{\boldsymbol{N}}_{\mathrm{BIE}} \\ \check{\boldsymbol{\xi}}_{\mathrm{BIE}} \end{pmatrix} = \arg\min_{\check{\boldsymbol{N}},\check{\boldsymbol{\xi}}} \mathrm{E}\left\{ \left\| \begin{pmatrix} \check{\boldsymbol{N}} \\ \check{\boldsymbol{\xi}} \end{pmatrix} - \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{\xi} \end{pmatrix} \right\|^2_{\boldsymbol{Q}^{-1}} \right\}.$$

Best Integer Equivariant Estimator:

$$\check{\boldsymbol{N}}_{\mathrm{BIE}} = \sum_{\boldsymbol{z}\in\mathbb{Z}^n} \boldsymbol{z} w_{\boldsymbol{z}}(\hat{\boldsymbol{N}}) \quad \text{with} \quad \sum_{\boldsymbol{z}\in\mathbb{Z}^n} w_{\boldsymbol{z}}\left(\hat{\boldsymbol{N}}\right) = 1$$

with the weights

$$w_{\boldsymbol{z}}\left(\hat{\boldsymbol{N}}\right) = \frac{\exp\left(-\frac{1}{2}\left\|\hat{\boldsymbol{N}} - \boldsymbol{z}\right\|^2_{\boldsymbol{Q}^{-1}_{\hat{\boldsymbol{N}}}}\right)}{\sum_{\boldsymbol{z}'\in\mathbb{Z}^n} \exp\left(-\frac{1}{2}\left\|\hat{\boldsymbol{N}} - \boldsymbol{z}'\right\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{N}}^{-1}}}\right)}$$

Best Integer Equivariant Estimation

Best Integer Equivariant Estimator:

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Benefits:

+ *optimal* MSE estimator

Major drawback:

- *search too complex*, not feasible for PPP

Sequential Best Integer Equivariant Estimation

Ambiguity fixing to *weighted sum* of *integer candidates*:

$$\check{N}_{l,\mathrm{SBIE}} = \sum_{z \in \mathbf{\Theta}^d_{\hat{N}_{l|\mathcal{L}}}} z\, w_z\left(\hat{N}_{l|\mathcal{L}}\right) \quad \forall\, l$$

which requires only a *one-dimensional search* with the search space

$$\mathbf{\Theta}^d_{\hat{N}_{l|\mathcal{L}}} = z \in \left[\hat{N}_{l|\mathcal{L}} - d\sigma_{\hat{N}_{l|\mathcal{L}}}, \hat{N}_{l|\mathcal{L}} + d\sigma_{\hat{N}_{l|\mathcal{L}}}\right] \quad \text{s.t.} \quad z \in \mathbb{Z}$$

and the weighting

$$w_z(\hat{N}_{l|\mathcal{L}}) = \frac{\exp\left(-\frac{1}{2}\left\|\hat{N}_{l|\mathcal{L}} - z\right\|^2_{Q^{-1}_{\hat{N}}}\right)}{\sum_{z' \in \mathbb{Z}^n} \exp\left(-\frac{1}{2}\left\|\hat{N}_{l|\mathcal{L}} - z'\right\|^2_{Q_{\hat{N}^{-1}}}\right)}.$$

Sequential Best Integer Equivariant Estimation

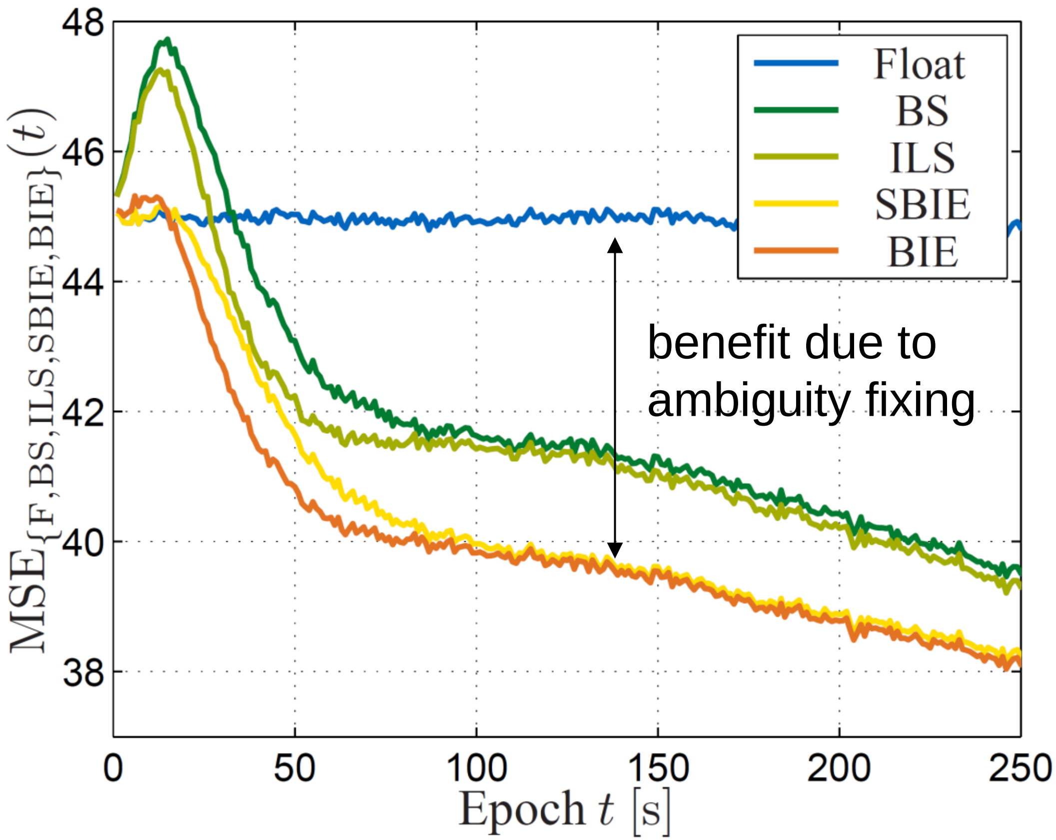
Ambiguity fixing to weighted sum of integer candidates:

$$\check{N}_{l,\mathrm{SBIE}} = \sum_{z \in \mathbf{\Theta}^{d}_{\hat{N}_{l|\mathcal{L}}}} z\, w_z\left(\hat{N}_{l|\mathcal{L}}\right) \quad \forall\, l$$

where the weighting is determined
with the sequential conditional estimates

$$\hat{N}_{j|\mathcal{J}} = \hat{N}_j - \sum_{l=1}^{j-1} \sigma_{\hat{N}_j \hat{N}_{l|\mathcal{L}}} \sigma^{-2}_{\hat{N}_l} \left(\hat{N}_{l|\mathcal{L}} - \check{N}_{l,\mathrm{SBIE}}\right) \quad \forall j \in \{1, \ldots, n\}$$

Sequential Best Integer Equivariant Estimation

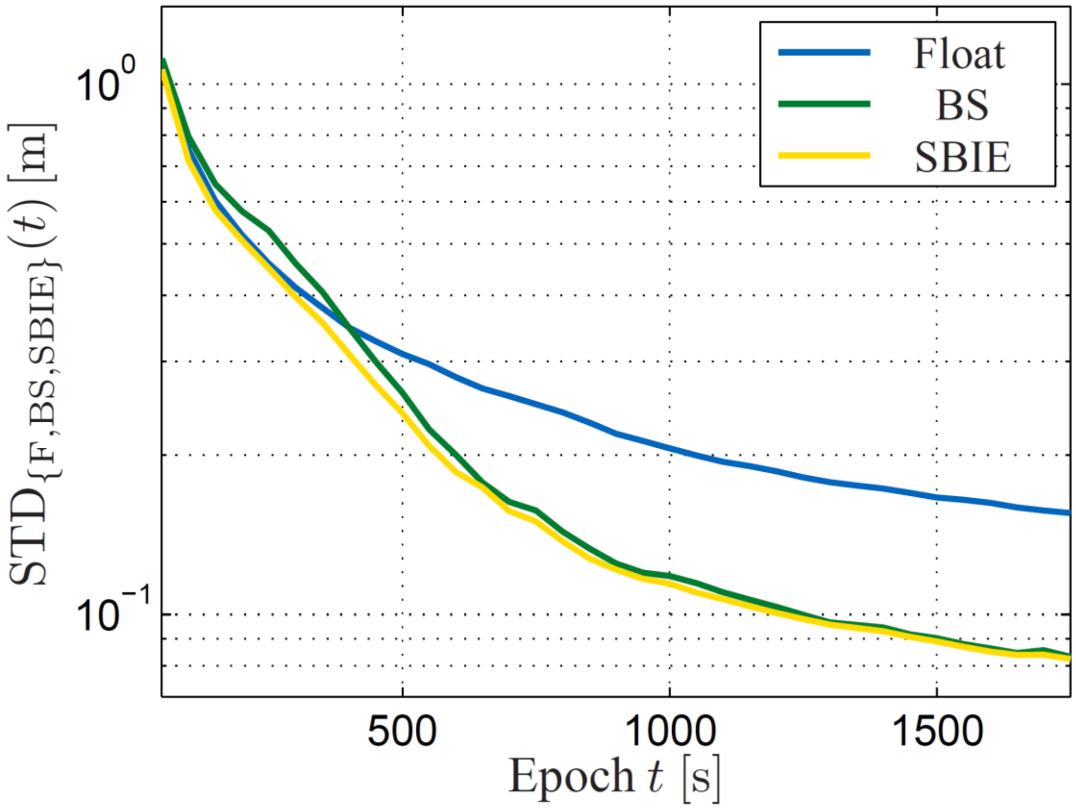


Geodetic network solution:

Estimation of
- ionospheric delays
- tropospheric zenith delay
- ambiguities
- receiver and satellite phase biases
- receiver and satellite clock offsets

[Source: Brack et al, ION ITM 2013]

Sequential Best Integer Equivariant Estimation



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Precise float ambiguity estimation: Multi-frequency combinations

Precise Point Positioning - measurement model:

$$\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \lambda\boldsymbol{\varphi}_1 \\ \lambda\boldsymbol{\varphi}_2 \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} \boldsymbol{x} \\ c\delta\tau \\ \boldsymbol{I} \\ T_{\mathrm{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \lambda_1 \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \lambda_2 \cdot \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\eta}_{\rho_1} \\ \boldsymbol{\eta}_{\rho_2} \\ \boldsymbol{\varepsilon}_{\lambda\varphi_1} \\ \boldsymbol{\varepsilon}_{\lambda\varphi_2} \end{bmatrix}$$

real-time and reliable integer ambiguity resolution is tough due to

- multipath
- satellite code and phase biases
- small carrier wavelengths => poor discrimination
- extremely ill-conditioned problem

Precise float ambiguity estimation: Multi-frequency combinations

Precise Point Positioning - measurement model:

$$\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \lambda\boldsymbol{\varphi}_1 \\ \lambda\boldsymbol{\varphi}_2 \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} \boldsymbol{x} \\ c\delta\tau \\ \boldsymbol{I} \\ T_{\mathrm{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \lambda_1 \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \lambda_2 \cdot \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\eta}_{\rho_1} \\ \boldsymbol{\eta}_{\rho_2} \\ \boldsymbol{\varepsilon}_{\lambda\varphi_1} \\ \boldsymbol{\varepsilon}_{\lambda\varphi_2} \end{bmatrix}$$

Mapping of integer ambiguities to reduced parameter space:

$$\{N_1^k, \ldots, N_M^k\} \overset{\longrightarrow}{\not\longleftarrow} N^k = \sum_{m=1}^{M} j_m N_m^k$$

Precise float ambiguity estimation: Multi-frequency combinations

Mapping of integer ambiguities to reduced parameter space:

$$\{N_1^k, \ldots, N_M^k\} \underset{\not\longleftarrow}{\longrightarrow} N^k = \sum_{m=1}^{M} j_m N_m^k$$

considered as one single vector!

$$\begin{bmatrix} \sum_{m=1}^{M} \alpha_m \lambda_m \boldsymbol{\varphi}_m + \beta_m \boldsymbol{\rho}_m \\ \sum_{m=1}^{M} \beta'_m \boldsymbol{\rho}_m \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} \boldsymbol{x} \\ c\delta\tau \\ T_z \end{bmatrix} + \begin{bmatrix} \lambda \cdot \mathbf{1} \\ \mathbf{0} \end{bmatrix} \overbrace{\left[\sum_{m=1}^{M} j_m \boldsymbol{N}_m \right]}^{=\boldsymbol{N}} + \begin{bmatrix} \sum_{m=1}^{M} \alpha_m \boldsymbol{\varepsilon}_{\lambda_m \varphi_m} + \beta_m \boldsymbol{\varepsilon}_{\rho_m} \\ \sum_{m=1}^{M} \beta'_m \boldsymbol{\eta}_{\rho_m} \end{bmatrix}$$

no slant ionospheric delays!

Precise float ambiguity estimation: Multi-frequency combinations

$$\sum_{m=1}^{M}(\alpha_m \lambda_m \phi_{u,m}^k + \beta_m \rho_{u,m}^k) = \left(\sum_{m=1}^{M}(\alpha_m + \beta_m)\right) \cdot \left(\|\boldsymbol{x}_u - \boldsymbol{x}^k\| + (\boldsymbol{e}_u^k)^T \delta \boldsymbol{x}^k + c(\delta\tau_u - \delta\tau^k) + T_u^k\right)$$
$$+ \left(\sum_{m=1}^{M}(\alpha_m - \beta_m) q_{1m}^2\right) \cdot {I'}_{u,1}^{k} + \left(\sum_{m=1}^{M}(\frac{1}{2}\alpha_m - \beta_m) q_{1m}^3\right) \cdot {I''}_{u,1}^{k}$$
$$+ \left(\sum_{m=1}^{M} \alpha_m \lambda_m N_m\right) + \left(\sum_{m=1}^{M} \alpha_m (b_{\phi_{u,m}} + b_{\phi_m^k}) + \beta_m (b_{\rho_{u,m}} + b_{\rho_m^k})\right)$$
$$+ \left(\sum_{m=1}^{M}(\alpha_m \ddot{o}_{\phi_{u,m}^k} + \beta_m \ddot{o}_{\rho_{u,m}^k})\right) + \left(\sum_{m=1}^{M}(\alpha_m \varepsilon_{\phi_{u,m}^k} + \beta_m \varepsilon_{\rho_{u,m}^k})\right)$$

(1) Geometry constraint: $$\sum_{m=1}^{M}(\alpha_m + \beta_m) = h_1$$

(2) Ionospheric delay (first order): $$\sum_{m=1}^{M}(\alpha_m - \beta_m) q_{1m}^2 = h_2$$

Precise float ambiguity estimation: Multi-frequency combinations

$$\sum_{m=1}^{M}(\alpha_m\lambda_m\phi_{u,m}^k+\beta_m\rho_{u,m}^k) = \left(\sum_{m=1}^{M}(\alpha_m+\beta_m)\right)\cdot\left(\|\boldsymbol{x}_u-\boldsymbol{x}^k\|+(\boldsymbol{e}_u^k)^T\delta\boldsymbol{x}^k+c(\delta\tau_u-\delta\tau^k)+T_u^k\right)$$
$$+\left(\sum_{m=1}^{M}(\alpha_m-\beta_m)q_{1m}^2\right)\cdot {I'}_{u,1}^k+\left(\sum_{m=1}^{M}(\frac{1}{2}\alpha_m-\beta_m)q_{1m}^3\right)\cdot {I''}_{u,1}^k$$
$$+\left(\sum_{m=1}^{M}\alpha_m\lambda_m N_m\right)+\left(\sum_{m=1}^{M}\alpha_m(b_{\phi_{u,m}}+b_{\phi_m^k})+\beta_m(b_{\rho_{u,m}}+b_{\rho_m^k})\right)$$
$$+\left(\sum_{m=1}^{M}(\alpha_m\ddot{o}_{\phi_{u,m}^k}+\beta_m\ddot{o}_{\rho_{u,m}^k})\right)+\left(\sum_{m=1}^{M}(\alpha_m\varepsilon_{\phi_{u,m}^k}+\beta_m\varepsilon_{\rho_{u,m}^k})\right)$$

(3) Integer ambiguities: $\sum_{m=1}^{M}\alpha_m\lambda_m N_m=\lambda N$ which is equivalent to $N=\sum_{m=1}^{M}\underbrace{\frac{\alpha_m\lambda_m}{\lambda}}_{=j_m}N_m$

$\Rightarrow \quad \alpha_m=\frac{j_m\lambda}{\lambda_m}$ with **any arbitrary** combination wavelenth $\lambda=\frac{w_\phi}{\sum_{m=1}^{M}\frac{j_m}{\lambda_m}}$

Precise float ambiguity estimation: Multi-frequency combinations

$$\begin{bmatrix} \sum_{m=1}^{M} \alpha_m \lambda_m \boldsymbol{\varphi}_m + \beta_m \boldsymbol{\rho}_m \\ \sum_{m=1}^{M} \beta'_m \boldsymbol{\rho}_m \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} \boldsymbol{x} \\ c\delta\tau \\ T_\mathrm{z} \end{bmatrix} + \begin{bmatrix} \lambda \cdot \mathbf{1} \\ \mathbf{0} \end{bmatrix} \overbrace{\left[\sum_{m=1}^{M} j_m \boldsymbol{N}_m \right]}^{=\boldsymbol{N}} + \begin{bmatrix} \sum_{m=1}^{M} \alpha_m \boldsymbol{\varepsilon}_{\lambda_m \boldsymbol{\varphi}_m} + \beta_m \boldsymbol{\varepsilon}_{\boldsymbol{\rho}_m} \\ \sum_{m=1}^{M} \beta'_m \boldsymbol{\eta}_{\boldsymbol{\rho}_m} \end{bmatrix}$$

How to choose the phase and code coefficients?

maximization of ambiguity discrimination $\dfrac{\lambda(\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M)}{2\sigma(\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M)}$

Precise float ambiguity estimation: Multi-frequency combinations

Geometric and ionospheric constraints:

$$\sum_{m=1}^{M} (\alpha_m + \beta_m) = h_1, \qquad \sum_{m=1}^{M} (\alpha_m - \beta_m) q_{1m}^2 = h_2$$

The phase coefficients are rewritten using the total phase weight defined as

$$w_\phi = \sum_{m=1}^{M} \alpha_m = \lambda \sum_{m=1}^{M} \frac{j_m}{\lambda_m} \quad \Rightarrow \lambda = \frac{w_\phi}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}}$$

and therefore

$$\alpha_m = \frac{j_m}{\lambda_m} \lambda = \frac{j_m}{\lambda_m} \frac{1}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}} w_\phi$$

Geometric/ ionospheric constraints in matrix-vector-notation: $\mathbf{\Psi}_1 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \mathbf{\Psi}_2 \begin{bmatrix} w_\phi \\ \beta_3 \\ \vdots \\ \beta_M \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

Precise float ambiguity estimation: Multi-frequency combinations

Geometric/ ionospheric constraints in matrix-vector-notation:

$$\mathbf{\Psi}_1 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \mathbf{\Psi}_2 \begin{bmatrix} w_\phi \\ \beta_3 \\ \vdots \\ \beta_M \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

Solving for the code weights $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ gives:

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{\Psi}_1^{-1} \left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - \mathbf{\Psi}_2 \begin{bmatrix} w_\phi \\ \beta_3 \\ \vdots \\ \beta_M \end{bmatrix} \right) = \begin{bmatrix} s_1 + s_2 w_\phi + \sum_{m=3}^{M} s_m \beta_m \\ t_1 + t_2 w_\phi + \sum_{m=3}^{M} t_m \beta_m \end{bmatrix}$$

Remaining unknowns:

$j_1, \ldots, j_M, w_\phi, \beta_3, \ldots, \beta_M$ used to maximize the ambiguity discrimination $D = \frac{\lambda}{2\sigma}$

Precise float ambiguity estimation: Multi-frequency combinations

Ambiguity discrimination: $D = \frac{\lambda}{2\sigma} = \frac{\lambda}{2\sqrt{\sum_{m=1}^{M} \alpha_m^2 \sigma_{\phi_m}^2 + \beta_m^2 \sigma_{\rho_m}^2}}$

where the wavelength and phase and code coefficients are replaced by

$$\lambda = \frac{w_\phi}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}}, \qquad \alpha_m = \frac{j_m}{\lambda_m}\lambda = \frac{j_m}{\lambda_m} \frac{1}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}} w_\phi$$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{\Psi}_1^{-1} \left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - \mathbf{\Psi}_2 \begin{bmatrix} w_\phi \\ \beta_3 \\ \vdots \\ \beta_M \end{bmatrix} \right) = \begin{bmatrix} s_1 + s_2 w_\phi + \sum_{m=3}^{M} s_m \beta_m \\ t_1 + t_2 w_\phi + \sum_{m=3}^{M} t_m \beta_m \end{bmatrix}$$

to obtain

$$D(w_\phi, \boldsymbol{\beta}) = \frac{\lambda}{2\sigma} = \frac{w_\phi}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}} \frac{1}{2\sqrt{\tilde{\eta}^2 w_\phi^2 + (s_1 + s_2 w_\phi + \boldsymbol{s}^T \boldsymbol{\beta})^2 \sigma_{\rho_1}^2 + (t_1 + t_2 w_\phi + \boldsymbol{t}^T \boldsymbol{\beta})^2 \sigma_{\rho_2}^2 + \boldsymbol{\beta}^T \mathbf{\Sigma} \boldsymbol{\beta}}}$$

Precise float ambiguity estimation: Multi-frequency combinations

Ambiguity discrimination:

$$D(w_\phi, \boldsymbol{\beta}) = \frac{\lambda}{2\sigma} = \frac{w_\phi}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}} \frac{1}{2\sqrt{\tilde{\eta}^2 w_\phi^2 + (s_1 + s_2 w_\phi + \boldsymbol{s}^T\boldsymbol{\beta})^2 \sigma_{\rho_1}^2 + (t_1 + t_2 w_\phi + \boldsymbol{t}^T\boldsymbol{\beta})^2 \sigma_{\rho_2}^2 + \boldsymbol{\beta}^T \boldsymbol{\Sigma} \boldsymbol{\beta}}}$$

Maximization over the total phase weight and code coefficients:

$$\frac{\partial D}{\partial w_\phi} \overset{!}{=} 0 \quad \text{and} \quad \frac{\partial D}{\partial \boldsymbol{\beta}} \overset{!}{=} \boldsymbol{0}$$

↓

$$\begin{aligned}
&(s_1 + s_2 w_\phi + \boldsymbol{s}^T\boldsymbol{\beta})\boldsymbol{s} \cdot \sigma_{\rho_1}^2 + (t_1 + t_2 w_\phi + \boldsymbol{t}^T\boldsymbol{\beta})\boldsymbol{t} \cdot \sigma_{\rho_2}^2 + \boldsymbol{\Sigma}\boldsymbol{\beta} \\
&= \sigma_{\rho_1}^2 \boldsymbol{s}(s_1 + s_2 w_\phi + \boldsymbol{s}^T\boldsymbol{\beta}) + \sigma_{\rho_2}^2 \boldsymbol{t}(t_1 + t_2 w_\phi + \boldsymbol{t}^T\boldsymbol{\beta}) + \boldsymbol{\Sigma}\boldsymbol{\beta} \\
&= \underbrace{\left[\sigma_{\rho_1}^2 \boldsymbol{s}\boldsymbol{s}^T + \sigma_{\rho_2}^2 \boldsymbol{t}\boldsymbol{t}^T + \boldsymbol{\Sigma}\right]}_{\boldsymbol{A}} \boldsymbol{\beta} + \underbrace{\left[s_2 \sigma_{\rho_1}^2 \boldsymbol{s} + t_2 \sigma_{\rho_2}^2 \boldsymbol{t}\right]}_{\boldsymbol{b}} w_\phi + \underbrace{\left[s_1 \sigma_{\rho_1}^2 \boldsymbol{s} + t_1 \sigma_{\rho_2}^2 \boldsymbol{t}\right]}_{\boldsymbol{c}} = \boldsymbol{0}.
\end{aligned}$$

Solving for $\boldsymbol{\beta}$ yields: $\boldsymbol{\beta} = -\boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b} \cdot w_\phi)$

Precise float ambiguity estimation: Multi-frequency combinations

Ambiguity discrimination:

$$D(w_\phi, \boldsymbol{\beta}) = \frac{\lambda}{2\sigma} = \frac{w_\phi}{\sum_{m=1}^{M} \frac{j_m}{\lambda_m}} \frac{1}{2\sqrt{\tilde{\eta}^2 w_\phi^2 + (s_1 + s_2 w_\phi + \boldsymbol{s}^T\boldsymbol{\beta})^2\sigma_{\rho_1}^2 + (t_1 + t_2 w_\phi + \boldsymbol{t}^T\boldsymbol{\beta})^2\sigma_{\rho_2}^2 + \boldsymbol{\beta}^T\boldsymbol{\Sigma}\boldsymbol{\beta}}}$$

Maximization over the total phase weight and code coefficients:

$$\frac{\partial D}{\partial w_\phi} \overset{!}{=} 0 \quad \text{and} \quad \frac{\partial D}{\partial \boldsymbol{\beta}} \overset{!}{=} \boldsymbol{0}$$

$$\begin{aligned}
&(s_1 + s_2 w_\phi + \boldsymbol{s}^T\boldsymbol{\beta})\boldsymbol{s}\cdot\sigma_{\rho_1}^2 + (t_1 + t_2 w_\phi + \boldsymbol{t}^T\boldsymbol{\beta})\boldsymbol{t}\cdot\sigma_{\rho_2}^2 + \boldsymbol{\Sigma}\boldsymbol{\beta} \\
&= \sigma_{\rho_1}^2\boldsymbol{s}(s_1 + s_2 w_\phi + \boldsymbol{s}^T\boldsymbol{\beta}) + \sigma_{\rho_2}^2\boldsymbol{t}(t_1 + t_2 w_\phi + \boldsymbol{t}^T\boldsymbol{\beta}) + \boldsymbol{\Sigma}\boldsymbol{\beta} \\
&= \underbrace{\left[\sigma_{\rho_1}^2\boldsymbol{s}\boldsymbol{s}^T + \sigma_{\rho_2}^2\boldsymbol{t}\boldsymbol{t}^T + \boldsymbol{\Sigma}\right]}_{\boldsymbol{A}}\boldsymbol{\beta} + \underbrace{\left[s_2\sigma_{\rho_1}^2\boldsymbol{s} + t_2\sigma_{\rho_2}^2\boldsymbol{t}\right]}_{\boldsymbol{b}} w_\phi + \underbrace{\left[s_1\sigma_{\rho_1}^2\boldsymbol{s} + t_1\sigma_{\rho_2}^2\boldsymbol{t}\right]}_{\boldsymbol{c}} = \boldsymbol{0}.
\end{aligned}$$

Solving for $\boldsymbol{\beta}$ yields: $\boldsymbol{\beta} = -\boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b}\cdot w_\phi)$

Precise float ambiguity estimation: Multi-frequency combinations

Maximization over the total phase weight:

$$\frac{\partial D}{\partial w_\phi} \overset{!}{=} 0$$

gives $\left(s_1 + s_2 w_\phi + \boldsymbol{s}^T \boldsymbol{\beta}\right)\left(s_1 + \boldsymbol{s}^T \boldsymbol{\beta}\right)\sigma_{\rho_1}^2 + \left(t_1 + t_2 w_\phi + \boldsymbol{t}^T \boldsymbol{\beta}\right)\left(t_1 + \boldsymbol{t}^T \boldsymbol{\beta}\right)\sigma_{\rho_2}^2 + \boldsymbol{\beta}^T \boldsymbol{\Sigma} \boldsymbol{\beta} = 0.$

and using $\boldsymbol{\beta} = -\boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b} \cdot w_\phi)$ yields

$$\begin{aligned}
&\left(s_1 + s_2 w_\phi - \boldsymbol{s}^T \boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b} w_\phi)\right) \cdot \left(s_1 - \boldsymbol{s}^T \boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b} w_\phi)\right) \cdot \sigma_{\rho_1}^2 \\
&+ \left(t_1 + t_2 w_\phi - \boldsymbol{t}^T \boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b} w_\phi)\right) \cdot \left(t_1 - \boldsymbol{t}^T \boldsymbol{A}^{-1}(\boldsymbol{c} + \boldsymbol{b} w_\phi)\right) \cdot \sigma_{\rho_2}^2 \\
&+ (\boldsymbol{c} + \boldsymbol{b} w_\phi)^T (\boldsymbol{A}^{-1})^T \boldsymbol{\Sigma} \boldsymbol{A}^{-1} (\boldsymbol{c} + \boldsymbol{b} w_\phi) = 0,
\end{aligned}$$

which is a quadratic equation in w_ϕ:

$$r_0 + r_1 \cdot w_\phi + r_2 \cdot w_\phi^2 = 0,$$

and can easily be solved for the optimal phase weighting.

Precise float ambiguity estimation: Multi-frequency combinations

Maximization of ambiguity discrimination

a. Analytical computation:

$$\max_{w_\phi, \boldsymbol{\beta}} D(\boldsymbol{j}, w_\phi, \boldsymbol{\beta})$$

b. Numerical search:

$$\max_{\boldsymbol{j}} D(\boldsymbol{j}, w_\phi, \boldsymbol{\beta})$$

Frequencies and noise assumptions:

- $f_1, \ldots, f_M$
- $\sigma_{\phi_1}, \ldots, \sigma_{\phi_M}, \sigma_{\rho_1}, \ldots, \sigma_{\rho_M}$
- Constraint on geometry: h_1
- Constraints on 1st and 2nd order ionospheric delays: h_2, h_3
- Constraints on biases and multipath

Outputs: $\{\alpha_1, \ldots, \alpha_M\}$ and $\{\beta_1, \ldots, \beta_M\}$

Precise float ambiguity estimation: Multi-frequency combinations

scaling of geometry

↑ ↗ scaling of ionospheric delay

enables reliable fixing!

h_1	h_2	E1		E5b		E5a		λ	σ	D
1	0	j_1	1	j_2	-4	j_3	3	3.603 m	13.9 cm	12.99
		α_1	18.9326	α_2	-58.0271	α_3	42.4139			
		β_1	-0.2871	β_2	-0.9899	β_3	-1.0423			
1	-0.1	j_1	1	j_2	-4	j_3	3	3.368 m	12.7 cm	13.26
		α_1	17.6991	α_2	-54.2465	α_3	39.6505			
		β_1	-0.2499	β_2	-0.9013	β_3	-0.9519			
0	-1	j_1	1	j_2	-4	j_3	3	2.543 m	12.3 cm	9.98
		α_1	-12.8901	α_2	39.5074	α_3	-28.8772			
		β_1	0.4477	β_2	0.9061	β_3	0.9061			
0	0	j_1	0	j_2	-1	j_3	1	1 m	0.8 cm	62.71
		α_1	0	α_2	-4.0266	α_3	3.9242			
		β_1	0.0004	β_2	0.0480	β_3	0.0540			

Precise float ambiguity estimation: Multi-frequency combinations

scaling of geometry

scaling of ionospheric delay

enables reliable fixing!

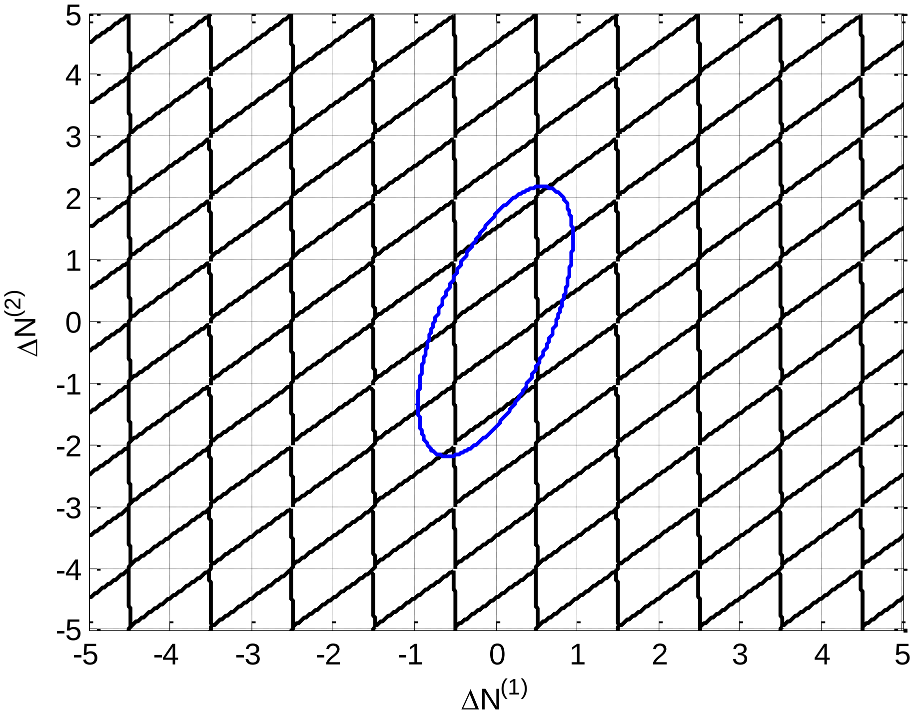
h_1	h_2	E1		E6		E5b		E5a		λ	σ	D
1	0	j_1	1	j_2	-3	j_3	0	j_4	2	3.998 m	6.5 cm	31.02
		α_1	21.0108	α_2	-51.1627	α_3	0	α_4	31.3798			
		β_1	-0.0239	β_2	-0.0349	β_3	-0.0824	β_4	-0.0867			
1	-0.1	j_1	1	j_2	-3	j_3	0	j_4	2	3.753 m	6.0 cm	31.22
		α_1	19.7197	α_2	-48.0187	α_3	0	α_4	29.4514			
		β_1	-0.0154	β_2	-0.0233	β_3	-0.0554	β_4	-0.0585			
0	-1	j_1	-1	j_2	4	j_3	-1	j_4	-2	2.505 m	5.1 cm	24.81
		α_1	-13.1658	α_2	42.7460	α_3	-10.0881	α_4	-19.6632			
		β_1	0.0285	β_2	0.0274	β_3	0.0576	β_4	0.0576			
0	0	j_1	0	j_2	0	j_3	-1	j_4	1	1 m	0.8 cm	64.27
		α_1	0	α_2	0	α_3	-4.0266	α_4	3.9242			
		β_1	-0.0038	β_2	0.0140	β_3	0.0429	β_4	0.0493			

small code coefficients

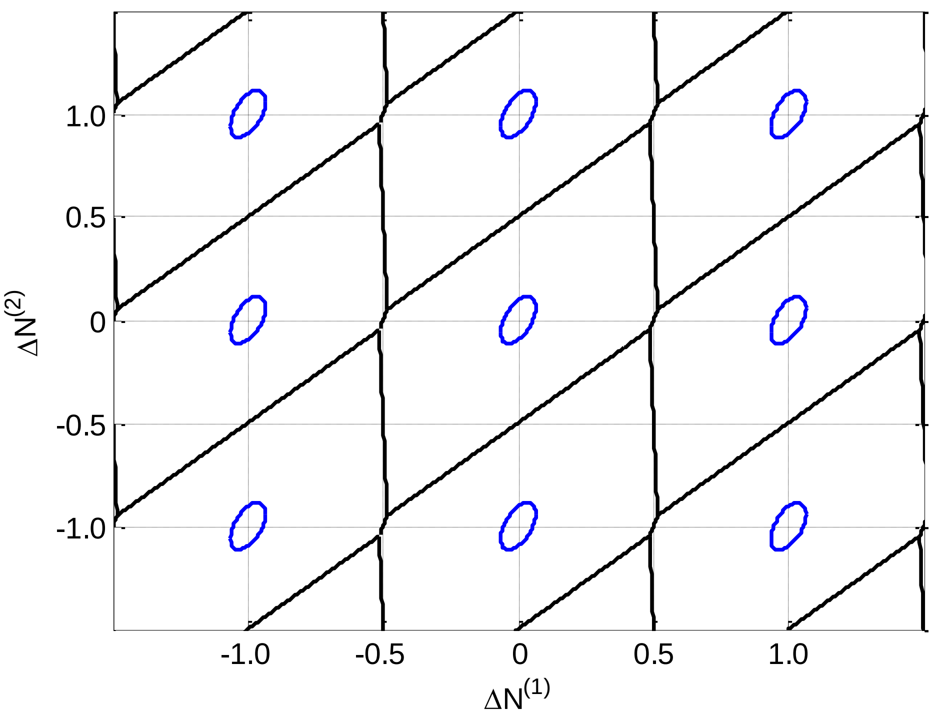
Precise float ambiguity estimation: Multi-frequency combinations

Integer-least-squares pull-in regions:

E1 pull-in regions:



Widelane pull-in regions with λ=3.285 m:



Precise float ambiguity estimation: Multi-frequency combinations

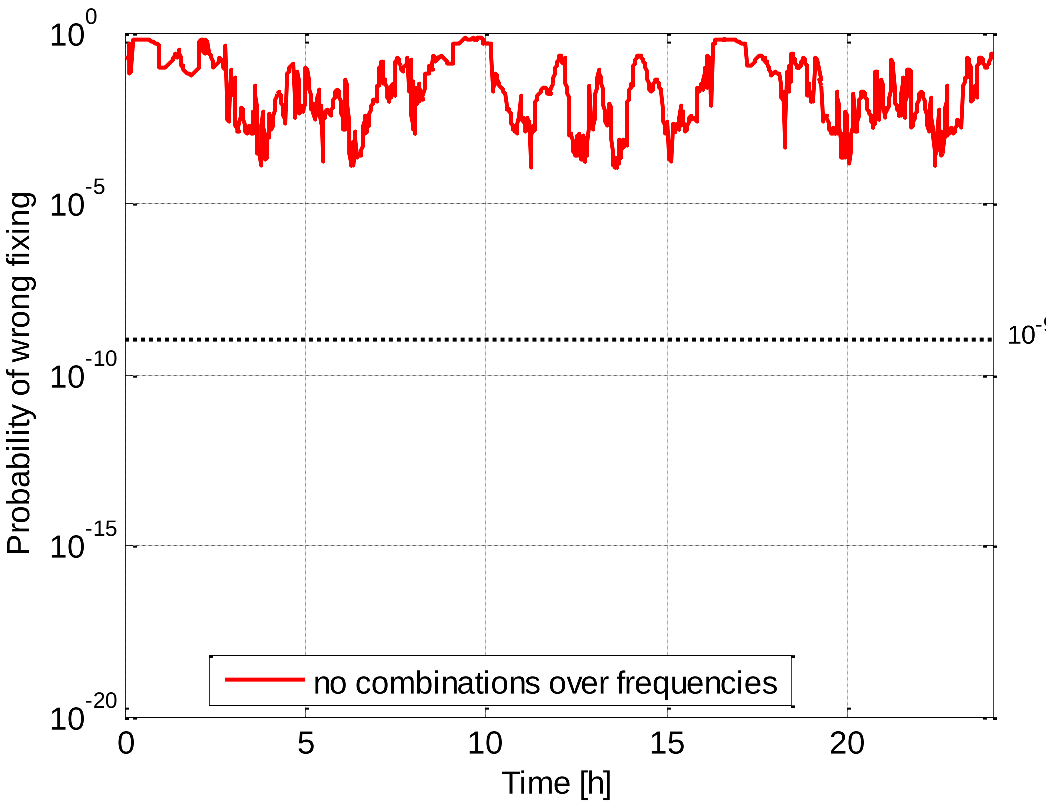
PPP using E1 and E5 code and carrier phase measurements of 3 s:

a.) Dual frequency (E1-E5) Galileo code and carrier phase measurements

without linear combinations

Estimated parameters:

- carrier phase integer ambiguities
- position (once/ epoch)
- ionospheric delays (initial epoch)
- gradients of ionospheric delays
- tropospheric wet zenith delay (initial epoch)
- gradient of trop. wet zenith delay



Precise float ambiguity estimation: Multi-frequency combinations

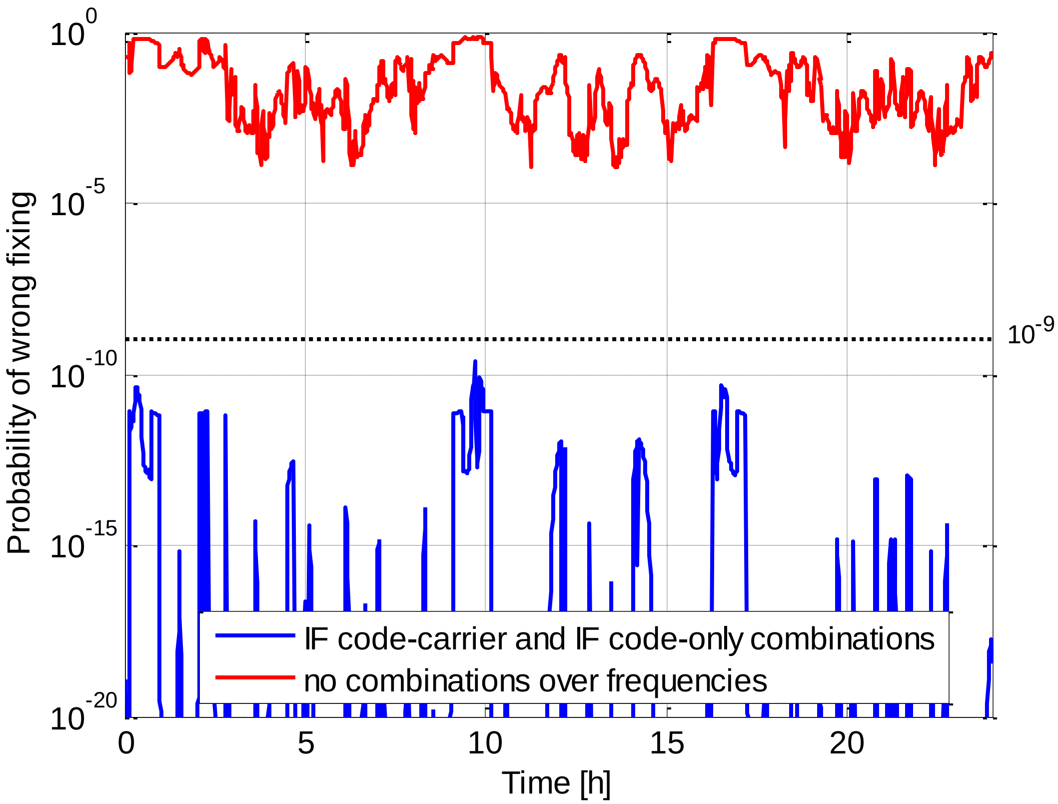
PPP using E1 and E5 code and carrier phase measurements of 3 s:

b.) Dual frequency (E1-E5) Galileo code and carrier phase measurements

with
two geometry-preserving, ionosphere-free
linear combinations:
- code carrier combination of maximum discrimination
- code-only combination

Estimated parameters:

- ♦ widelane integer ambiguities
- ♦ position (once/ epoch)
- ♦ tropospheric wet zenith delay and its gradient



Integer ambiguity resolution with real Galileo signals

Two geodetic OEM 628 Novatel receivers:

- tracking of following Galileo signals:

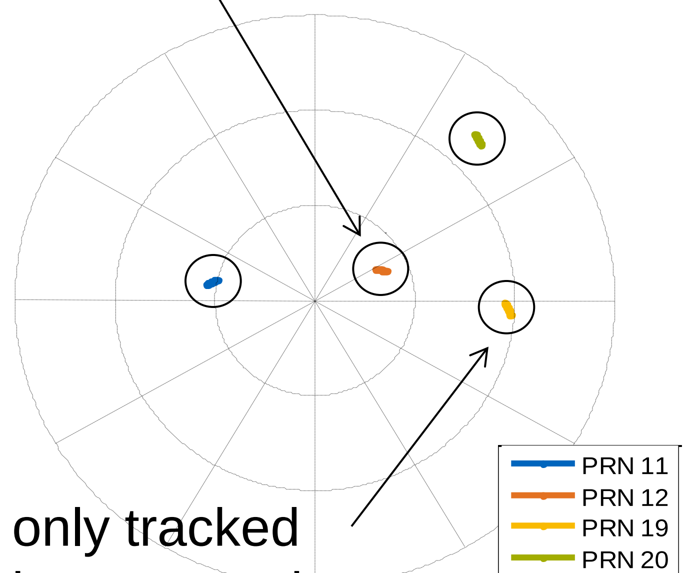
E1:	C1C	L1C	D1C	S1C
E5a:	C5Q	L5Q	D5Q	S5Q
E5b:	C7Q	L7Q	D7Q	S7Q
E5:	C8Q	L8Q	L8Q	S8Q

- tracking also of L1/L2/L5 GPS signals

baseline length:

$$l = 1.25\text{m}$$

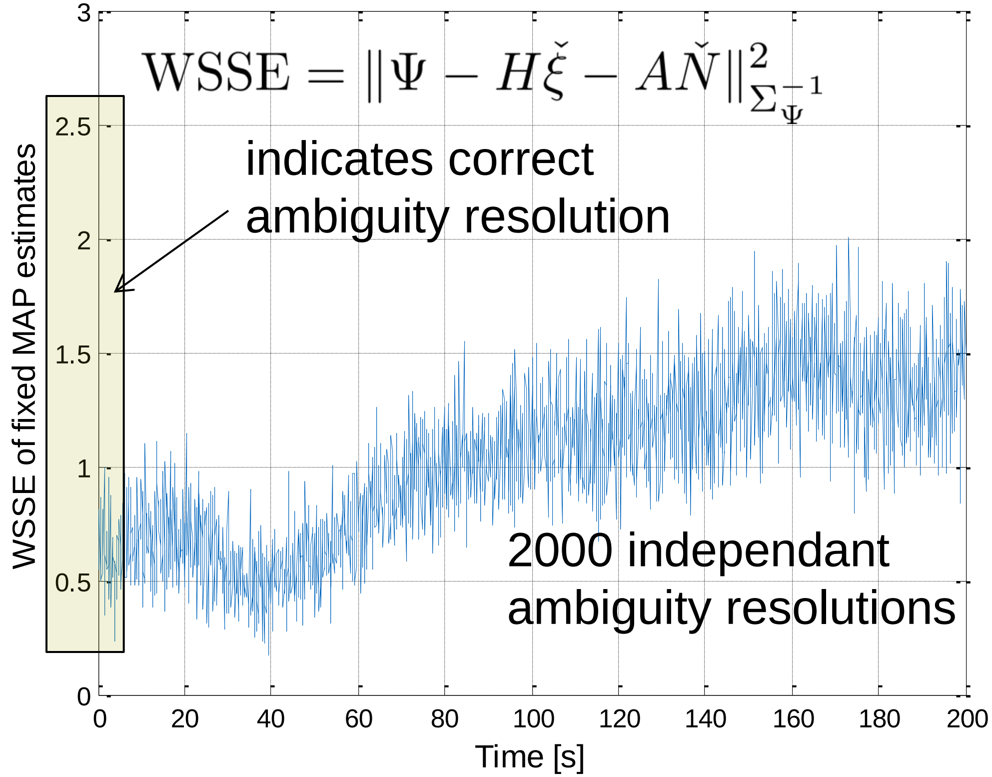
reference satellite



only tracked by one receiver

Differential integer ambiguity resolution

- ***Single epoch*** E1-E5a ***widelane*** integer ambiguity resolution for 1.25 m baseline with baseline length and height a priori information

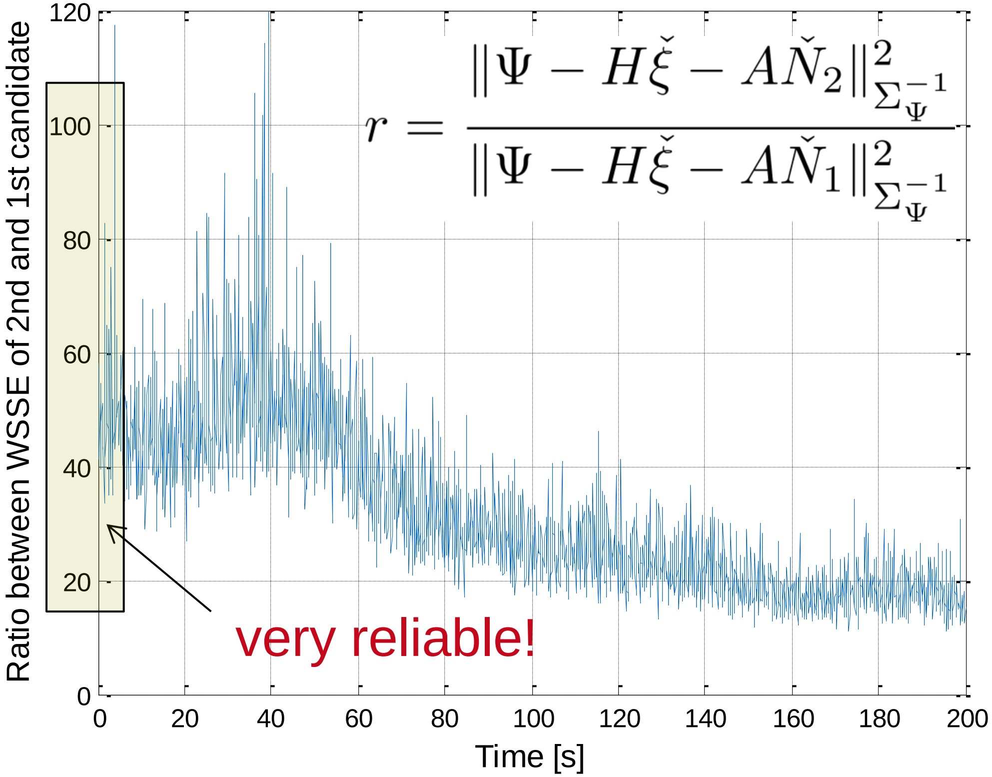


baseline length:

$$l = 1.25\mathrm{m}$$

Differential integer ambiguity resolution

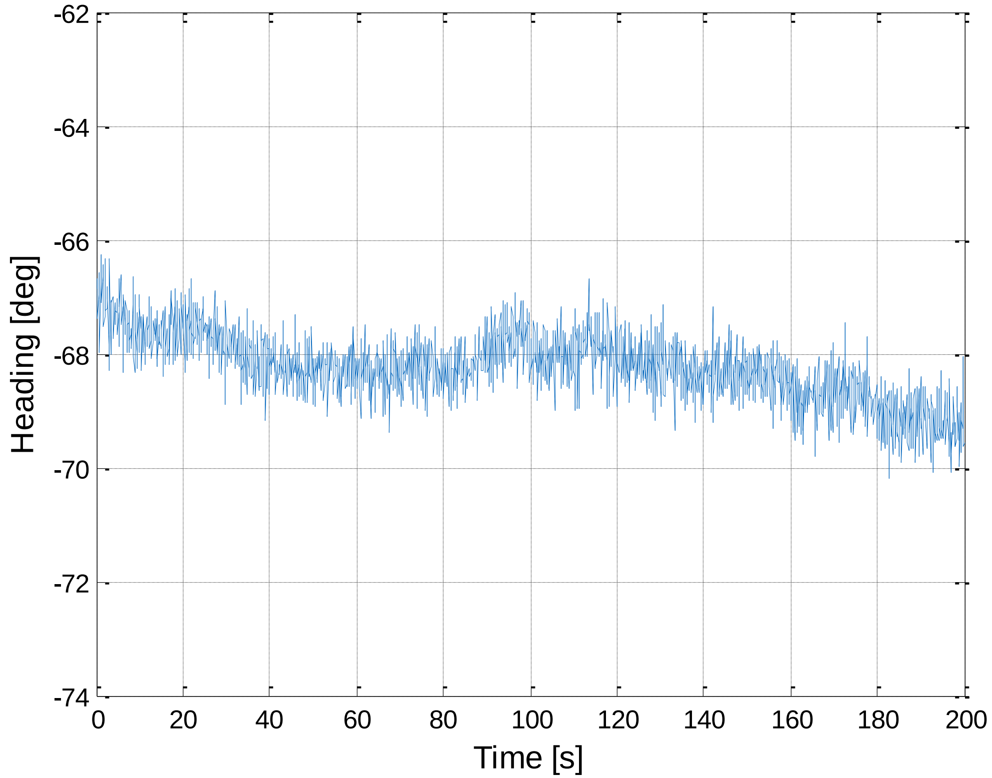
- ***Single epoch*** E1-E5a ***widelane*** integer ambiguity resolution for 1.25 m baseline with baseline length and height a priori information

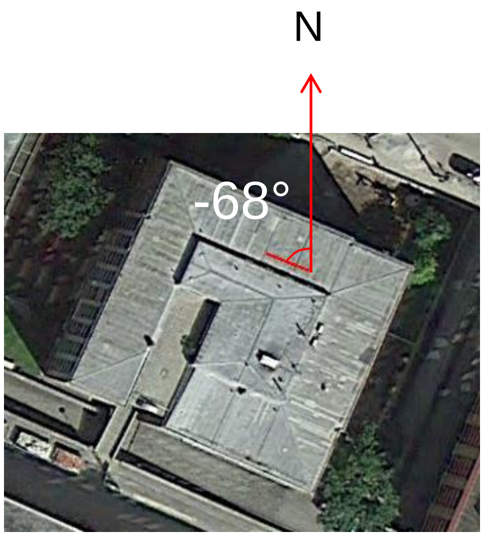


Note: Each of the 2000 independant ambiguity resolutions (10 Hz meas. rate) was correct.

Heading determination

➢ Heading determination based on fixed widelane ambiguities:
$\sigma_{\text{head}} = 0.6^\circ$ (corresponds to sigma = 1.3 cm for relative position)



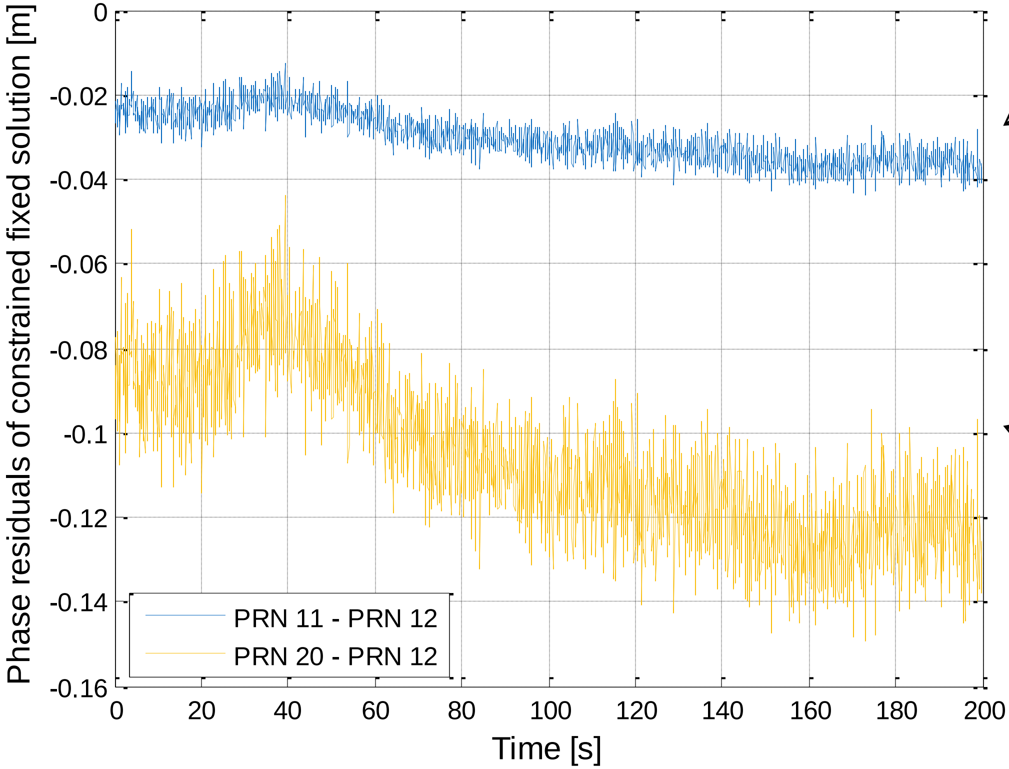


baseline length:

$l = 1.25\text{m}$

Phase residuals of constrained fixed solution

- ***Single epoch*** E1-E5a ***widelane*** integer ambiguity resolution for 1.25 m baseline with baseline length and height a priori information



$E \in \{56.9^\circ, 57.9^\circ\}$

$E \in \{19.9^\circ, 19.5^\circ\}$

multipath

Conclusion

- Introduction to Sequential Best Integer Equivariant Estimation attractive to Precise Point Positioning as it

  + enables a lower Mean Square Error (MSE) than the conventional LAMBDA method
  + requires only *n* one-dimensional searches instead of one *n*-dimensional search and
  + lower computational complexity than LAMBDA

- Overview of multi-frequency linear combinations for Galileo

  + enable arbitrary scaling of geometry
  + enable arbitrary scaling of ionospheric delay
  + enable an desired wavelength

- Single epoch integer ambiguity resolution with discrimination of 100.