Do we need a conventional transformation model for vertical reference frames?

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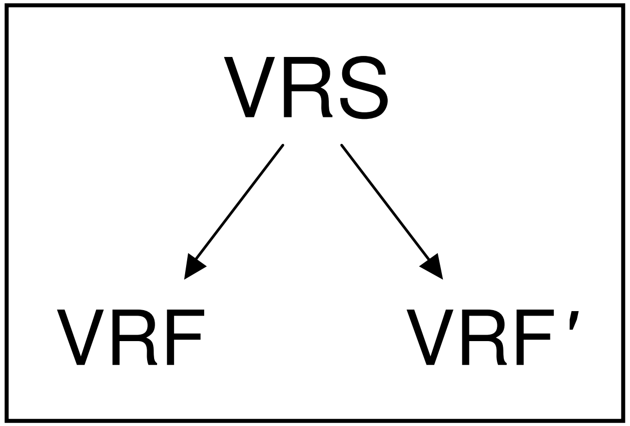
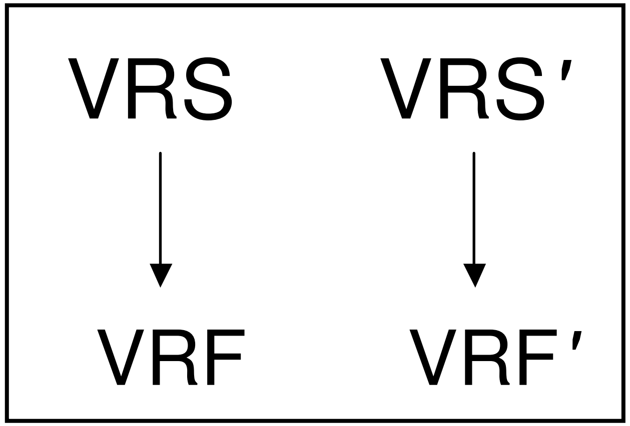
Introduction

The basic concept..



or



How much two different realizations of the same or different vertical reference systems differ from each other ?

Introduction

Conventional comparison of **3D spatial TRFs**
(linearized form of similarity transformation)

$$\begin{bmatrix} x'-x \\ y'-y \\ z'-z \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} + \begin{bmatrix} \delta s & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & \delta s & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & \delta s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Conventional comparison of **1D spatial VRFs**
for physically meaningful heights

$$H' - H = f(H, \textit{datum perturbation parameters})$$

?

Common height transformations

- *Corrector surfaces* for GPS-aided leveling within a local vertical datum

$$h - N - H^{LVD} = \mathbf{a}^T\mathbf{x} + s + v$$

- Estimation of Earth's *mean equatorial radius* and *CoM* from heterogeneous height data

$$N(h, H) - N(C_{nm}, S_{nm}, \Delta g) = f(\delta a, \delta f, t_x, t_y, t_z)$$

- Other auxiliary transformations
  (e.g. change of tidal system, normal-to-ortho height conversion, reduction due to modeled geodynamic effects, etc.)

However…

- A conventional transformation model for different VRFs is **not** presently in use
- It should employ specific parameters to quantify the (actual + apparent) inconsistencies in the realization of 1D vertical reference systems
- Why is it needed ?
    - ….
    - ….
    - ….

basically, for the same reasons that the conventional 3D similarity transformation is useful in spatial TRF studies (*more details to follow*)

Datum perturbation parameters

	TRF → TRF′	VRF → VRF′
Shift	t_x, t_y, t_z	δW_o
Rotation	$\varepsilon_x, \varepsilon_y, \varepsilon_z$	–
Scale	δs	δs (*)

- The TRF scale change factor is not equivalent with the VRF scale change factor !

Forward effect of δW_o

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o + \delta W_o$
Geopot. number	$c(P_i)$	$c(P_i) + \delta W_o$
Orthometric height	$H(P_i)$	$H(P_i) + \frac{\delta W_o}{g_i} - \frac{1}{2}\frac{\partial g}{\partial H}\frac{\delta W_o^2}{g_i^3} + \ldots$
Normal height	$\tilde{H}(P_i)$	$\tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i} - \frac{1}{2}\frac{\partial \gamma}{\partial h}\frac{\delta W_o^2}{\gamma_i^3} + \ldots$

Orthometric and normal heights are affected in a **nonlinear** and **spatially inhomogeneous** way by δW_o

Forward effect of δW_o

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o + \delta W_o$
Geopot. number	$c(P_i)$	$c(P_i) + \delta W_o$
Orthometric height	$H(P_i)$	$H(P_i) + \frac{\delta W_o}{g_i} - \frac{1}{2}\frac{\partial g}{\partial H}\frac{\delta W_o^2}{g_i^3} + \ldots$
Normal height	$\tilde{H}(P_i)$	$\tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i} - \frac{1}{2}\frac{\partial \gamma}{\partial h}\frac{\delta W_o^2}{\gamma_i^3} + \ldots$

The contribution of the second (and higher) order terms is **negligible** (< 1 mm) even for δW_o up to 100 $m^2\, s^{-2}$

Forward effect of δW_o

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o + \delta W_o$
Geopot. number	$c(P_i)$	$c(P_i) + \delta W_o$
Orthometric height	$H(P_i)$	$H(P_i) + \frac{\delta W_o}{g_i} - \frac{1}{2}\frac{\partial g}{\partial H}\frac{\delta W_o^2}{g_i^3} + \ldots$ (g_i → γ_i)
Normal height	$\tilde{H}(P_i)$	$\tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i} - \frac{1}{2}\frac{\partial \gamma}{\partial h}\frac{\delta W_o^2}{\gamma_i^3} + \ldots$

Replacing g_i by γ_i causes a **negligible error** (< 1 mm) in the transformed orthometric height when $|\delta W_o| < 20\ \mathrm{m^2\,s^{-2}}$, even for $\Delta g = g_i - \gamma_i = 500$ mgal

Conventional modeling

Rigorous form (for geopotential numbers)

$$c'(P_i) = c(P_i) + \delta W_o$$

Semi-rigorous form (for normal heights)

$$\tilde{H}'(P_i) = \tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i}$$

Approximate form (for orthometric heights)

- "small δW_o approximation"
- consistent at the mm-level for $|\delta W_o| < 20\ \mathrm{m^2\,s^{-2}}$

$$H'(P_i) = H(P_i) + \frac{\delta W_o}{\gamma_i}$$

Forward effect of δs

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o$
Geopotential number difference	$\Delta c_{ij} = c(P_j) - c(P_i)$	$\Delta c'_{ij} = (1+\delta s) \cdot \Delta c_{ij}$
Orthometric height difference	$\Delta H_{ij} = H(P_j) - H(P_i)$	$\Delta H'_{ij} = (1+\delta s) \cdot \Delta H_{ij}$
Normal height difference	$\Delta \tilde{H}_{ij} = \tilde{H}(P_j) - \tilde{H}(P_i)$	$\Delta \tilde{H}'_{ij} = (1+\delta s) \cdot \Delta \tilde{H}_{ij}$

Uniform **scale change** along a certain spatial direction that is used for physical height determination

(*) with respect to a fixed reference surface

Conventional modeling

$$c'(P_i) = c(P_i) + \delta s \cdot c(P_i)$$ *Geopotential numbers*

$$H'(P_i) = H(P_i) + \delta s \cdot H(P_i)$$ *Orthometric heights*

$$\tilde{H}'(P_i) = \tilde{H}(P_i) + \delta s \cdot \tilde{H}(P_i)$$ *Normal heights*

- Zero-height points are preserved
- The scale change factor *(δs)* **is not identical** among the various height types !
- A scale factor is an ideal tool to describe (the linear part of) **topographically-correlated discrepancies** among different VRFs

Conventional VRF transformation

Combined effect of “origin” and “scale” change:

$$c'(P_i) = (1+\delta s)\cdot c(P_i) \ + \ \delta W_o$$

$$H'(P_i) = (1+\delta s)\cdot H(P_i) \ + \ \frac{\delta W_o}{\gamma_i}$$

$$\tilde{H}'(P_i) = (1+\delta s)\cdot \tilde{H}(P_i) \ + \ \frac{\delta W_o}{\gamma_i}$$

Should we use the above conventional models to infer VRF inconsistencies over a terrestrial network ?

Optimal LS inversion

Given two realizations VRF (**d**) and VRF′(**d**′),
and a weight matrix **P** for their differences, the relative 'datum perturbations' can be jointly estimated as:

$$\begin{bmatrix} \delta\hat{W}_o \\ \delta\hat{s} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T\mathbf{P}\mathbf{q} & \mathbf{q}^T\mathbf{P}\mathbf{d} \\ \mathbf{d}^T\mathbf{P}\mathbf{q} & \mathbf{d}^T\mathbf{P}\mathbf{d} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \\ \mathbf{d}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \end{bmatrix}$$

(∗) If **d** & **d**′ are geopotential numbers

$$\mathbf{q}^T = [1 \quad \cdots \quad 1]$$

(∗) If **d** & **d**′ are orthometric or normal heights

$$\mathbf{q}^T = [1/\gamma_1 \quad \cdots \quad 1/\gamma_N]$$

Optimal LS inversion

Given two realizations VRF (**d**) and VRF′(**d**′), and a weight matrix **P** for their differences, the relative ‘datum perturbations’ can be jointly estimated as:

$$\begin{bmatrix} \delta\hat{W}_o \\ \delta\hat{s} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{q}^T\mathbf{P}\mathbf{q} & \mathbf{q}^T\mathbf{P}\mathbf{d} \\ \mathbf{d}^T\mathbf{P}\mathbf{q} & \mathbf{d}^T\mathbf{P}\mathbf{d} \end{bmatrix}}^{-1} \begin{bmatrix} \mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \\ \mathbf{d}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \end{bmatrix}$$

↓

Invertible matrix, provided that: □ $(\mathbf{d},\mathbf{q}) \neq 0$

Example

Network	d	d'	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
20 EUVN_DA points (Switzerland)	EVRF00	EVRF07	0.025	2.9
	GPS/EGG08	EVRF07	0.044	-76.6
	GPS/EGG97	EVRF07	-0.159	-107.7
	LN02	EVRF07	-0.251	35.7
	LHN95	EVRF07	-0.060	-220.7
22 EUVN_DA points (Hungary)	GPS/EGG08	EVRF07	-0.035	-110.9

Residuals (mean/std)

Network	d	d′	Before transformation (cm)	After transformation (cm)
20 EUVN_DA points (Switzerland)	EVRF00	EVRF07	2.8 / 0.3	0.0 / 0.2
	GPS/EGG08	EVRF07	-3.2 / 5.2	0.0 / 2.6
	GPS/EGG97	EVRF07	-27.1 / 9.9	0.0 / 7.6
	LN02	EVRF07	-22.0 / 6.9	0.0 / 6.6
	LHN95	EVRF07	-28.3 / 14.0	0.0 / 5.6
22 EUVN_DA points (Hungary)	GPS/EGG08	EVRF07	-6.1 / 2.8	0.0 / 2.3

Another example

Network	d	d′	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
13 'core datum' UELN points over EU (see Sacher et al.)	EVRF00	EVRF07	0.002	-25.5

It should be zero (theoretically) !

The effect of LS correlation…

$$\begin{bmatrix} \delta\hat{W}_o \\ \delta\hat{s} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T\mathbf{P}\mathbf{q} & \mathbf{q}^T\mathbf{P}\mathbf{d} \\ \mathbf{d}^T\mathbf{P}\mathbf{q} & \mathbf{d}^T\mathbf{P}\mathbf{d} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \\ \mathbf{d}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \end{bmatrix}$$

Equivalently,

$$\delta\hat{W}_o = \boxed{\frac{\mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d})}{\mathbf{q}^T\mathbf{P}\mathbf{q}}} + \rho_{\delta\hat{W}_o,\delta\hat{s}} \left(\frac{\mathbf{d}^T\mathbf{P}\mathbf{d}}{\mathbf{q}^T\mathbf{P}\mathbf{q}}\right)^{1/2} \delta\hat{s}$$

$$\rho_{\delta\hat{W}_o,\delta\hat{s}} = -\frac{\mathbf{q}^T\mathbf{P}\mathbf{d}}{(\mathbf{d}^T\mathbf{P}\mathbf{d})^{1/2}\cdot(\mathbf{q}^T\mathbf{P}\mathbf{q})^{1/2}} \approx -\frac{mean\,[\mathbf{d}]}{rms\,[\mathbf{d}]}$$

Alternative δs-estimation scheme

Based on the **height differences** that can be formed **within each frame** (for a number of VRF baselines) using a suitable selection matrix **B**

$$\delta\hat{s} = \frac{\mathbf{d}^T\mathbf{B}^T\mathbf{P}^*\mathbf{B}(\mathbf{d}'-\mathbf{d})}{\mathbf{d}^T\mathbf{B}^T\mathbf{P}^*\mathbf{B}\mathbf{d}}$$

P*: weight matrix for the double differences **B**(**d**′ – **d**)

- Selection of VRF baselines
- Choice of weight matrix **P***
- δW_o can be estimated after reducing **d** and **d**′ to a common spatial scale

Example

Network	**d**	**d′**	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
13 ‘core datum’ UELN points over EU (see Sacher et al.)	EVRF00	EVRF07	0.002	-25.5

Alternative ‘sequential estimation scheme’

From independent baselines with $\mathbf{P}^* = (\mathbf{B}\mathbf{P}^{-1}\mathbf{B}^T)^{-1}$	0.002	-25.5
From independent baselines with $\mathbf{P}^* = \mathbf{I}$	0.002	-16.9
From independent baselines with $\mathbf{P}^* = f\,(1/L_{ij})$	0.001	-13.6
From all baselines with $\mathbf{P}^* = \mathbf{I}$	0.002	-25.5
From all baselines with $\mathbf{P}^* = f\,(1/L_{ij})$	0.003	-27.2

Example

Network	**d**	**d′**	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
20 EUVN_DA points (Switzerland)	LN02	EVRF07	-0.251	35.7

Alternative 'sequential estimation scheme'

From independent baselines with **P*** = $(\mathbf{B}\mathbf{P}^{-1}\mathbf{B}^{T})^{-1}$	-0.251	35.7
From independent baselines with **P*** = **I**	-0.293	78.7
From independent baselines with **P*** $= f\,(1/L_{ij})$	-0.295	80.1
From all baselines with **P*** = **I**	-0.251	35.7
From all baselines with **P*** $= f\,(1/L_{ij})$	-0.241	25.6

Example

Network	d	d′	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
22 EUVN_DA points (Hungary)	GPS/EGG08	EVRF07	-0.035	-110.9

Alternative 'sequential estimation scheme'

From independent baselines with **P*** = $(\mathbf{B}\mathbf{P}^{-1}\mathbf{B}^{T})^{-1}$	-0.035	-110.9
From independent baselines with **P*** = **I**	-0.039	-92.0
From independent baselines with **P*** $= f\,(1/L_{ij})$	-0.031	-131.4
From all baselines with **P*** = **I**	-0.035	-110.9
From all baselines with **P*** $= f\,(1/L_{ij})$	-0.037	-104.1

Vertical S-transformation

Forward

$$\mathbf{d} = \mathbf{d}^o + \begin{bmatrix} \mathbf{q} & \mathbf{d}^o \end{bmatrix} \begin{bmatrix} \delta W_o \\ \delta s \end{bmatrix}$$

$$\mathbf{x} = \mathbf{x}^o + \mathbf{E}^T \boldsymbol{\theta}$$

Inverse

$$\hat{\boldsymbol{\theta}} = (\mathbf{E}\mathbf{E}^T)^{-1} \mathbf{E}(\mathbf{x} - \mathbf{x}^o)$$

$$\begin{bmatrix} \delta \hat{W}_o \\ \delta \hat{s} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T \mathbf{q} & \mathbf{q}^T \mathbf{d}^o \\ \mathbf{d}^{oT} \mathbf{q} & \mathbf{d}^{oT} \mathbf{d}^o \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}^T (\mathbf{d} - \mathbf{d}^o) \\ \mathbf{d}^{oT} (\mathbf{d} - \mathbf{d}^o) \end{bmatrix}$$

Important tool $\rightarrow$ full or partial inner constraints

(*) development of an optimal VRF from heterogeneous data sources (e.g. leveling, GPS/geoid, tide-gauge data, etc.)

Summary

- δW_o and δs are the basic conventional VRF transformation parameters (for static cases)
- Useful for evaluating the spatial consistency between different VRFs
  (additional distortion modeling may be also needed)
- A conventional VRF transformation provides the basis for vertical datum definition in cases of heterogeneous height data
- Generalization to time-dependent problems is necessary (temporal evolution of a VRF)

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