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# A Rigorous Formula For Geoid-to-Quasigeoid Separation

by

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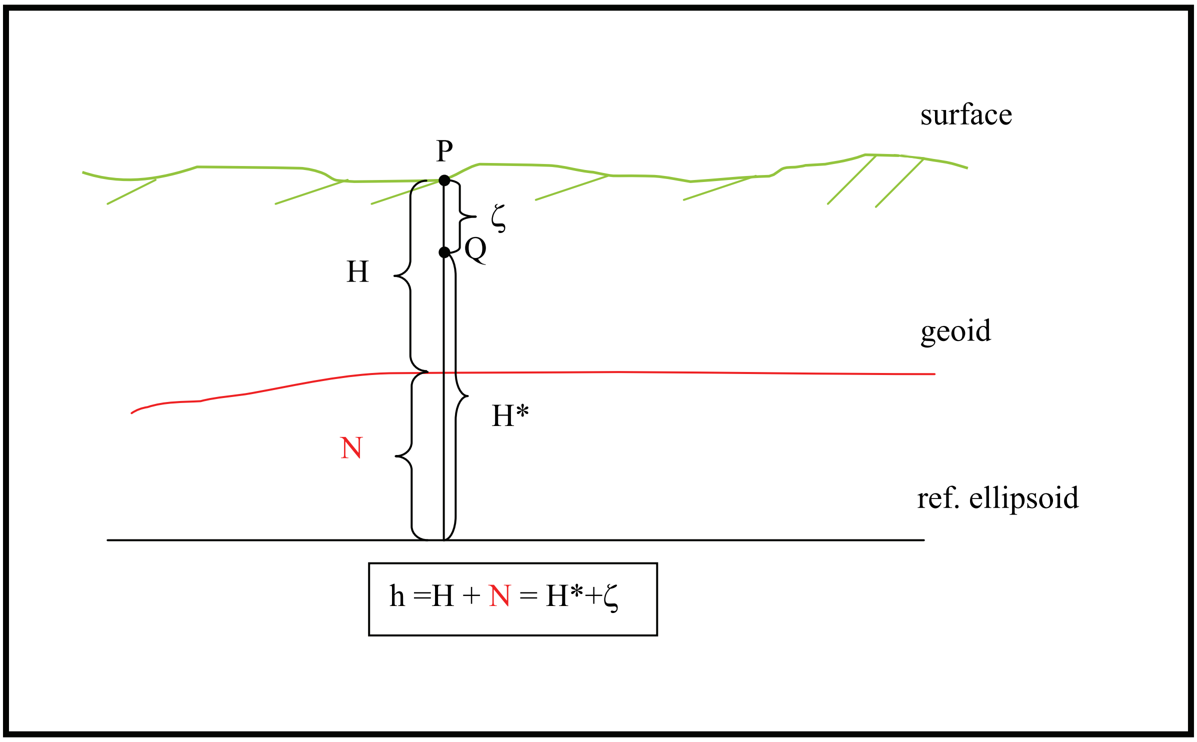
## CONTENTS


surface
P
ζ
Q
H
geoid
H*
N
ref. ellipsoid
h =H + N = H*+ζ

# 1 Introduction

**The traditional formula:**

$$N - \zeta = H^* - H = \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H, \qquad (1)$$

**where $\bar{g}$ and $\bar{\gamma}$ are the mean gravity between the geoid and Earth's surface and mean normal gravity between the reference ellipsoid and normal height. Also**

$N - \zeta$ = **geoid-quasigeoid heights,**

**and**

$H^* - H =$ **normal height - orthometric height.**

**Problem: Convert (1) to a more practical form.**

**The standard practical formula:**

$$N - \zeta \approx \frac{\Delta g_P^B}{\bar{\gamma}} H, \qquad (2)$$

where $\Delta g_P^B$ = **(simple) Bouguer gravity anomaly**.

**How can we improve Eq. (2)?**

# 2 Basic assumptions

## Assumptions and notations:

### 1. Bruns' formula

$$\zeta = \frac{T}{\gamma} \tag{3}$$

**holds for any point $P'$ between the geoid and the Earth's surface (including these surfaces),**

**$T$ = disturbing potential**
**$\gamma$ = normal gravity at normal height vs. $P'$.**

## Examples:

$$N = \frac{T_g}{\gamma_0} \quad \text{= geoid height} \tag{4a}$$

and

$$\zeta_P = \frac{T_P}{\gamma_Q} \quad \text{= quasigeoid height} \tag{4b}$$

**where $T_g$ and $T_P$ are the disturbing potentials at the geoid and surface point P, respectively, and $\gamma_0$ and $\gamma_Q$ are the normal gravity values at the reference ellipsoid and normal height, respectively**

2. **The no-topography and topographical quasigeoid heights are defined by**

$$\zeta^{nt} = \frac{T - V^t}{\gamma} = \frac{T_g^{nt}}{\gamma} \quad \textbf{and} \quad \zeta^t = \frac{V^t}{\gamma}, \tag{5}$$

**where $\gamma$ is located at normal height vs. the position of $T$ and $V^t$. $T^{nt}$ is the no-topography disturbing potential.**

3. **The quasigeoid height, the disturbing potential and the gravity anomaly (all denoted with superscript *i= 0)* can be decomposed into the no-topography (*i* = *nt*) and topographic (*i*= *t*) components, e.g.**

$$\zeta = \zeta^0 = \zeta^{nt} + \zeta^t . \tag{6}$$

4. **The gravity anomalies are related to the disturbing potentials according the approximation provided by the "boundary condition"(cf. Heiskanen and Moritz 1967, p. 85 )**

$$\Delta g^j = -\frac{\partial T^j}{\partial h} + \frac{T^j}{\gamma}\frac{\partial \gamma}{\partial h}, \quad j = 0, nt, t. \tag{7}$$

**where h is the height along the normal to the reference ellipsoid.**

**Note 1. The no-topography gravity anomaly differs from the refined Bouguer gravity anomaly as follows:**

$$\Delta g^{nt} = \Delta g^{BO} - \frac{V^t}{\gamma}\frac{\partial \gamma}{\partial h} \tag{8a}$$

where

$$\Delta g^{BO} = \Delta g - A^t \ , \quad \text{with} \quad A^t = -\frac{\partial V^t}{\partial h} . \tag{8b}$$

**Note 2. Eqs. (3) and (7) imply that**

$$\left(\frac{\partial \zeta^j}{\partial h}\right) = \frac{\partial}{\partial h}\left(\frac{T^j}{\gamma}\right) = -\left(\frac{\Delta g^j}{\gamma}\right); \quad j = 0, nt, t. \tag{9}$$

## 3 The main results

***Proposition 1*:**

$$N - \zeta_P = \int_0^{H_P} \frac{\Delta g}{\gamma} dh . \tag{10}$$

***Proof*: The proposition follows directly from Eqs. (9):**

$$\mathbf{N} - \zeta_{\mathbf{P}} = \int_{H_P}^{0} \frac{\partial \zeta}{\partial \mathbf{h}} \mathbf{dh} = \int_{0}^{\mathbf{H_P}} \frac{\Delta \mathbf{g}}{\gamma} \mathbf{dh}\,. \tag{11}$$

***Proposition 2*:**

$$\mathbf{N} - \zeta_{\mathbf{P}} = \frac{\Delta \mathbf{g}_{\mathbf{P}}^{\mathbf{BO}}}{\bar{\gamma}} \mathbf{H}_{\mathbf{P}} + \mathbf{TC} + \mathbf{GC}\,, \tag{12a}$$

**where**

$$\mathbf{TC} = \underbrace{\frac{\mathbf{V}_{\mathbf{g}}^{t}}{\gamma_{0}} - \frac{\mathbf{V}_{\mathbf{P}}^{t}}{\gamma_{\mathbf{Q}}}}_{TC1} - \frac{\mathbf{V}_{\mathbf{P}}^{t} \mathbf{H}_{\mathbf{P}}}{\gamma_{\mathbf{Q}} \bar{\gamma}} \frac{\partial \gamma}{\partial \mathbf{h}}\,, \tag{12b}$$

**(topographic correction)**

**and**

$$\mathbf{GC} = \int_{0}^{\mathbf{H_P}} \left( \frac{\Delta \mathbf{g}^{\mathbf{nt}}}{\gamma} - \frac{\Delta \mathbf{g}_{\mathbf{P}}^{\mathbf{nt}}}{\bar{\gamma}} \right) \mathbf{dh}\,. \tag{12c}$$

**(gravimetric correction)**

***Proof*: The proof is given below.**

## Proof of Proposition 2

**By introducing the notation**

$$TC1 = \frac{V_g^t}{\gamma_0} - \frac{V_P^t}{\gamma_Q} , \qquad (P1)$$

**and noting that**

$$\frac{\Delta g^{nt}}{\bar{\gamma}} = \frac{\Delta g^{BO}}{\bar{\gamma}} - \frac{V_P^t H_P}{\gamma_Q \bar{\gamma}} \frac{\partial \gamma}{\partial h} , \qquad (P2)$$

**it follows from Prop. 1 that the right member of the proposition can be written**

$$RM = TC1 + \int_0^{H_P} \frac{\Delta g^{nt}}{\gamma} dh = TC1 + \int_{H_p}^{0} \frac{\partial}{\partial h} (\zeta^{nt}) dh = TC1 + N^{nt} - \zeta_P^{nt} = N - \zeta_P , \qquad (P3)$$

**and the proposition follows.**

# 4 The topographic and gravimetric corrections

## *4.1 The topographic correction*

- **The topographic correction can be written**

$$TC = T\overline{C} + dTC \, , \tag{13a}$$

**where**

$$\overline{TC} = \frac{V_g^t - V_P^t}{\overline{\gamma}} \tag{13b}$$

**and**

$$dTC = V_g^t\left(\frac{1}{\gamma_0} - \frac{1}{\overline{\gamma}}\right) - V_P^t\left(\frac{1}{\gamma_Q} - \frac{1}{\overline{\gamma}}\right) - \frac{V_P^t H_P}{\gamma_Q \overline{\gamma}} \frac{\partial \gamma}{\partial h} \approx \frac{V_g^t - V_P^t}{2\overline{\gamma}^2} H_P \frac{\partial \gamma}{\partial h} \, . \tag{13c}$$

**Considering the spherical approximation (cf. Heiskanen and Moritz 1967, p. 87)**

$$\frac{1}{\overline{\gamma}} \frac{\partial \gamma}{\partial h} \approx -\frac{2}{R} \, , \tag{14}$$

**where R is sea-level radius, Eq. (13c) becomes**

$$dTC \approx -\frac{V_g^t - V_P^t}{\overline{\gamma}} \frac{H_P}{R} \, . \tag{15}$$

**As** $H_P / R < 1.4x10^{-3}$**,** ƆΤƀ **can be safely neglected in practical applications of Eq. (13a), and the topographic correction thus agrees with that of Flury and Rummel (2009.)**

- **Decomposing the topographic potential into the potentials of a Bouguer plate and a terrain effect** $\left(dV^t\right)$**, and noting that the Bouguer plate potential is the same on top as at the bottom of the plate, Eq. (13b) becomes**

$$\overline{TC} = \frac{dV_g^t - dV_P^t}{\overline{\gamma}}, \tag{16}$$

**i.e. the topographic effect reduces to the difference between the terrain effects at the geoid and the surface point.**

## *4.2 The gravimetric correction*

**The strict gravimetric correction, Eq. (11c), can be rewritten and developed into a Taylor series as follows:**

$$GC = -\int_0^{H_P}\left[\frac{\partial\zeta^{nt}}{\partial h}+\frac{\Delta g_P^{nt}}{\bar{\gamma}}\right]dh = \zeta_g^{nt}-\zeta_P^{nt}-\frac{\Delta g_P^{nt}}{\bar{\gamma}}H_P =$$
$$\sum_{k=2}^{\infty}\frac{H_P^k}{k!\,\gamma_Q}(-1)^k\left(\frac{\partial\zeta^{nt}}{\partial h}\right)_P^k+\Delta g_P^{nt}H_P\left(\frac{1}{\gamma_Q}-\frac{1}{\bar{\gamma}}\right)\approx -\frac{H_P^2}{2\gamma_Q}\left(\frac{\partial\Delta g^{nt}}{\partial h}\right)_P, \qquad (17)$$

**where we have truncated the series to its first term**

## 5 Conclusions

- **The strict formula was given by Proposition 2.**

- **A practical formula reads**

$$\mathbf{N - \zeta \approx \frac{\Delta g_P^{BO}}{\bar{\gamma}} H_P + \frac{dV_g^t - dV_P^t}{\bar{\gamma}} - \frac{H_P^2}{2\gamma_Q}\left(\frac{\partial \Delta g^{nt}}{\partial h}\right)_P}, \qquad (18)$$

**several m + several dm + several cm(?)**

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# THANK YOU!