The International Terrestrial Reference System and ETRS89: Part I : General concepts

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OUTLINE

- Concept and Definition
- What is a Terrestrial Reference System (TRS), how is it realized ?
- TRS Realization by a Frame (TRF)
- Reference frame representations for a deformable Earth

- Combination of TRF solutions
- Stacking of TRF time series

Defining a Reference System & Frame:

Three main conceptual levels :

- **Ideal Terrestrial Reference System (TRS):**
  Ideal, mathematical, theoretical system (not accessible)
- **Terrestrial Reference Frame (TRF):**
  Numerical realization of the TRS to which users have access
- **Coordinate System: cartesian (X,Y,Z), geographic (λ, ϕ, h), …**

– **The TRF is a materialization of the TRS inheriting the mathematical properties of the TRS**

– **As the TRS, the TRF has an origin, scale & orientation**

– **TRF is constructed using space geodesy observations, hence with uncertainties**

Ideal Terrestrial Reference System

A tridimensional reference frame (mathematical sense)
Defined in an Euclidian affine space of dimension 3:

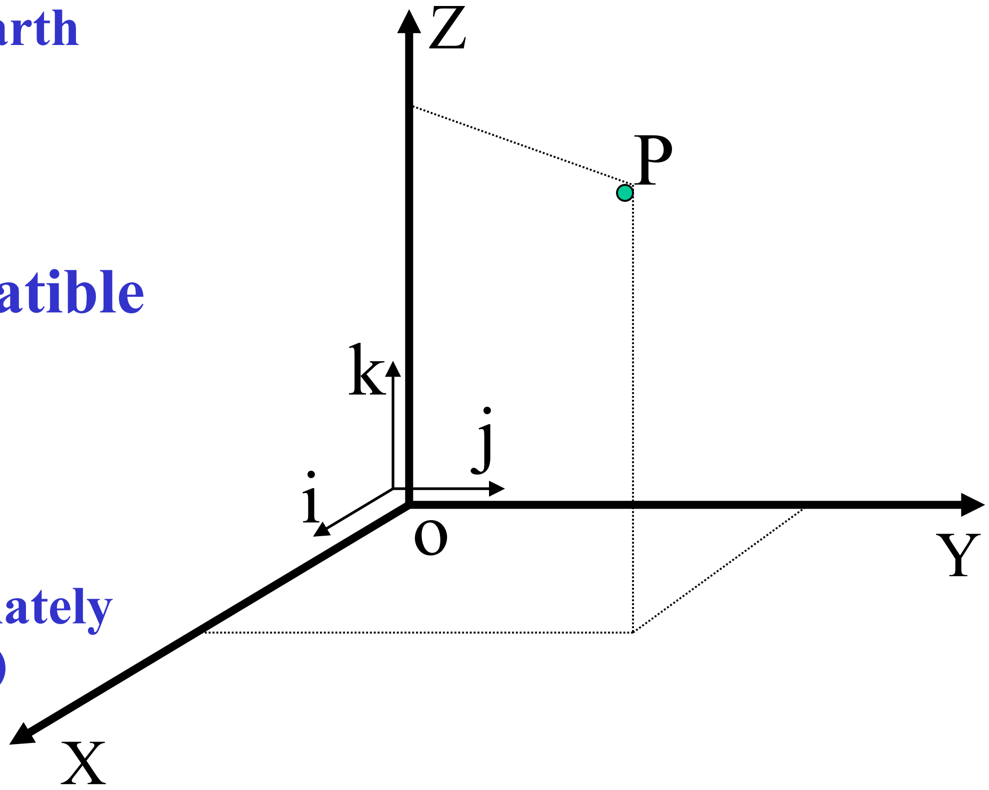
Affine Frame (O,E) where:

O: point in space (Origin)
E: vector base: orthogonal with the same length:
- **vectors co-linear to the base (Orientation)**
- **unit of length (Scale)**

$$\lambda = \left\| \vec{E}_i \right\|_{i=1,2,3} \qquad \vec{E}_i . \vec{E}_j = \lambda^2 \delta_{ij} \qquad (\delta_{ij} = 1, \quad i = j)$$

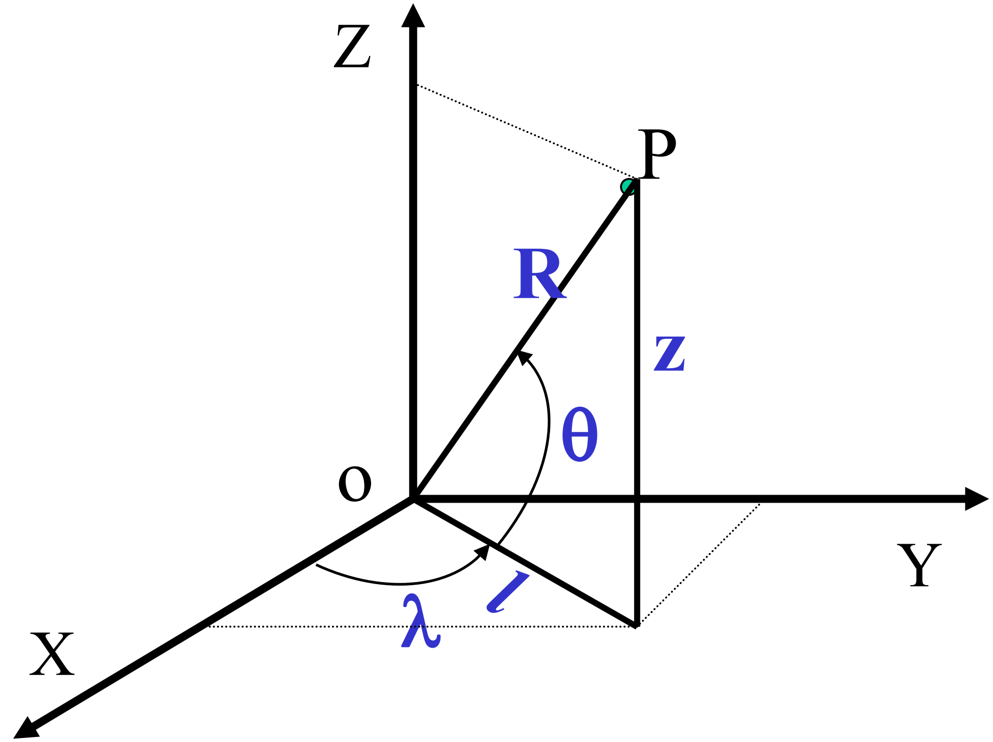
Terrestrial Reference Frame in the context of space geodesy

- **Origin:**
  - Center of mass of the whole Earth System, including oceans & atmosphere
- **Scale (unit of length): Compatible with SI unit**
- **Orientation:**
  - Equatorial (Z axis is approximately the direction of the Earth pole)



Coordinate Systems

- Cartesian: X, Y, Z
- Ellipsoidal: λ, φ, h
- Mapping: E, N, h
- Spherical: R, θ, λ
- Cylindrical: *l*, λ, Z



$$OP \left| \begin{array}{l} l\cos\lambda \\ l\sin\lambda \\ z \end{array} \right.$$

Cylindrical

$$OP \left| \begin{array}{l} R\cos\theta\cos\lambda \\ R\cos\theta\sin\lambda \\ R\sin\theta \end{array} \right.$$

Spherical

Transformation between TRS (1/2)

7-parameter similarity :

$$\boxed{X_2 = T + \lambda.\mathfrak{R}.X_1}$$

Translation Vector $T = (T_x, T_y, T_z)^T$

Scale Factor λ

Rotation Matrix $\mathfrak{R} = R_x.R_y.R_z$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos R1 & \sin R1 \\ 0 & -\sin R1 & \cos R1 \end{pmatrix}$$

$$R_y = \begin{pmatrix} \cos R2 & 0 & -\sin R2 \\ 0 & 1 & 0 \\ \sin R2 & 0 & \cos R2 \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos R3 & \sin R3 & 0 \\ -\sin R3 & \cos R3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformation between TRS (2/2)

In the context of space geodesy we use the linearized formula:

$$X_2 = X_1 + T + DX_1 + R.X_1 \qquad (1)$$

with: $T = (T_x, T_y, T_z)^T$, $\lambda = (1 + D)$, and $\mathcal{R} = (I + R)$

where $R = \begin{pmatrix} 0 & -R3 & R2 \\ R3 & 0 & -R1 \\ -R2 & R1 & 0 \end{pmatrix}$

T is less than 1 meter, D and R are less than 10^{-8}

The terms of 2nd ordre are neglected. They are less than $10^{-16} \ll 0.01$ mm.

Differentiating equation 1 with respect to time, we have:

$$\dot{X}_2 = \dot{X}_1 + \dot{T} + \overbrace{D\dot{X}_1}^{\approx 0} + \dot{D}X_1 + \overbrace{R\dot{X}_1}^{\approx 0} + \dot{R}X_1 \qquad (2)$$

"Motions" of the deformable Earth

- **Nearly linear motion:**
  - Tectonic motion: mainly horizontal
  - Post-Galcial Rebound: Vertical & Horizontal (or Glacial Isostatic Adjustment-GIA)

- **Non-Linear motion:**
  - Seasonal: Annual, Semi & Inter-Annual caused by loading effects
  - Rupture or dislocation: caused by EQ, Volcano Eruptions, etc.
  - Post-seismic deformation

Current/possible Reference Frame Representations

- "Quasi-Instantaneous" Frame: average of station positions over a "short" time-span:
  - One day or one week of observations

==> Non-linear motion embedded in time series of quasi-instantaneous reference frames

- Long-Term Secular Frame: mean station positions at a reference epoch (t_0) and station velocities: $\boxed{X(t) = X(t_0) + \dot{X}(t - t_0)}$

Crust-based TRF

The instantaneous position of a point on Earth Crust at epoch t could be written as :

$$X(t) = X_0 + \dot{X}.(t - t_0) + \sum_i \Delta X_i(t)$$

X_0 : **point position at a reference epoch t_0**

$\dot{X}$: **point linear velocity**

$\Delta X_i(t)$: **high frequency time variations:**

- **Solid Earth, Ocean & Pole tides**
- **Loading effects: atmosphere, ocean, hydrology, Post-glacial-Rebound**
- **... Earthquakes**

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Implementation of a TRF (1/2)

- Using raw observations of one or more space geodetic techniques, and one software package
- Stacking/cumulating a time series of station positions
- Combining several TRF solutions provided by different techniques (ITRF-like combination)
  - Needs local ties between technique reference points at co-location sites
  - If long-term solutions to be combined, need to equate velocities of multiple stations at co-location sites

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Implementation of a TRF (2/2)

- **Definition at a given epoch: specifying**

  7 parameters/components: 3 origin, 1 scale & 3 orientation

- **Time evolution: 7 rates of the 7 defining parameters, assuming linear station motion!**

- **==> 14 parameters are needed to define a TRF**

How to define the 14 parameters ?
« TRF definition »

- **Origin & rate:** CoM (Satellite Techniques)
- **Scale & rate:** depends on physical parameters
- **Orientation:** conventional
- **Orient. Rate:** conventional: Geophysical meaning (Tectonic Plate Motion)

- ==> Lack of information for some parameters:
  - Orientation & rate (all techniques)
  - Origin & rate in case of VLBI
  - ==> Rank Deficiency in terms of Normal Eq. System

Implmentation of a TRF in practice

The normal equation constructed upon observations of space techniques is written in the form of:

$$N.(\Delta X) = K \qquad (1)$$

where $\Delta X = X_{est} - X_{apr}$ are the linearized unknowns

Eq. (1) is a singular system: has a rank deficiency equal to the number of TRF parameters not given by the observations. Additional constraints are needed:

- Tight constraints ($\sigma \leq 10^{-10}$) m
- Removable constraints ($\sigma \cong 10^{-5}$) m
- Loose constraints ($\sigma \geq 1$) m

Applied over station coordinates

$$\boxed{(X_{est} - X_{apr}) = 0 \qquad (\sigma)}$$

- Minimum constraints (applied over the TRF parameters, see next)

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TRF definition using minimum constraints (1/3)

The standard relation linking two TRFs 1 and 2 is:

$$\boxed{X_2 = X_1 + A\theta}$$

$$X_i = (x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)^T$$

$$\theta = (T1, T2, T3, D, R1, R2, R3, \dot{T}1, \dot{T}2, \dot{T}3, \dot{D}, \dot{R}1, \dot{R}2, \dot{R}3)^T$$

θ **is the vector of the 7 (14) transformation parameters**

Least squares adjustment gives for ***θ*** **:**

$$\boxed{\theta = \overbrace{(A^T A)^{-1} A^T}^{\mathbf{B}} (X_2 - X_1)}$$

A : **desigin matrix of partial derivatives given in the next slide**

The Design matrix A

14 parameters

7 parameters

$$A = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 0 & 0 & x_i^0 & 0 & z_i^0 & -y_i^0 & & & & & & & \\
0 & 1 & 0 & y_i^0 & -z_i^0 & 0 & x_i^0 & & & & 0 & & & \\
0 & 0 & 1 & z_i^0 & y_i^0 & -x_i^0 & 0 & & & & & & & \\
 & & & & & & & 1 & 0 & 0 & x_i^0 & 0 & z_i^0 & -y_i^0 \\
 & & & \approx 0 & & & & 0 & 1 & 0 & y_i^0 & -z_i^0 & 0 & x_i^0 \\
 & & & & & & & 0 & 0 & 1 & z_i^0 & y_i^0 & -x_i^0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

Note: A could be reduced to specific parameters, e.g. if only rotations and rotation rates are needed, then the first 4 columns of the two parts of A are deleted

TRF definition using minimum constraints (2/3)

- The equation of minimum constraints is written as:

$$\boxed{B(X_2 - X_1) = 0 \qquad\qquad (\Sigma_\theta)}$$

It nullifies the 7 (14) transformation parameters between TRF 1 and TRF 2 at the Σ_θ level

- The normal equation form is written as:

$$\boxed{B^T \Sigma_\theta^{-1} B(X_2 - X_1) = 0}$$

Σ_θ is a diagonal matrix containing small variances of the 7(14) parameters, usually at the level of 0.1 mm

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TRF definition using minimum constraints (3/3)

Considering the normal equation of space geodesy:

$$N_{nc}(\Delta X) = K \qquad (1)$$

where $\Delta X = X_{est} - X_{apr}$ are the linearized unknowns

Selecting a reference solution X_R ,the equation of minimal constraints is given by:

$$B^T \Sigma_\theta^{-1} B(\Delta X) = B^T \Sigma_\theta^{-1} B(X_R - X_{apr}) \qquad (2)$$

Accumulating (1) and (2), we have:

$$\boxed{(N_{nc} + B^T \Sigma_\theta^{-1} B)(\Delta X) = K + B^T \Sigma_\theta^{-1} B(X_R - X_{apr})}$$

Note: if $X_R = X_{apr}$, the 2nd term of the right-hand side vanishes

Combination of TRF solutions

Combination of daily or weekly TRF solutions (1/3)

The basic combination model is written as:

$$\boxed{X_s^i = X_c^i + T_s + D_s X_c^i + R_s X_c^i}$$

Inputs: X_s^i, coordinates of point i of individual solution s.
Outputs (unknowns): combined coordinates X_c^i and transformation parameters T_s, D_s, R_s from TRF s to TRF c.
Note that the translation vector T_s and the rotation matrix R_s have each three components around the three axes X, Y, Z.

The unknown parameters are linearized around their approximate values: x_0^i, y_0^i, z_0^i, so that $x_c^i = x_0^i + \delta x^i$ (respectively y_c^i, z_c^i).

Note: this combination model is valid at a give epoch, *t_s*, both for the input and output station coordinates

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Combination of daily or weekly TRF solutions (2/3)

The observation equation system is written as:

$$\begin{pmatrix} I & A_s \end{pmatrix} \begin{pmatrix} \delta\chi_s \\ \delta T_s \end{pmatrix} + B_s = V_s$$

and the normal equation is:

$$\begin{pmatrix} P_s & P_s A_s \\ A_s^T P_s & A_s^T P_s A_s \end{pmatrix} \begin{pmatrix} \delta\chi_s \\ \delta T_s \end{pmatrix} + \begin{pmatrix} P_s B_s \\ A_s^T P_s B_s \end{pmatrix} = 0$$

where I is the identity matrix, As is the design matrix related to solution s, $\delta\chi_s$ and δT_s are the linearized unknowns of station coordinates and transformation parameters, respectively. B_s are the (observed - computed) values and V_s are the residuals. P_s: weight matrix = Σ_s^{-1}: inverse of variance-covariance matrix.

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Combination of daily or weekly TRF solutions (3/3)

The design matrixs A_s has the following form:

$$A_s = \begin{pmatrix} . & . & . & . & . & . & . \\ 1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i \\ 0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i \\ 0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 \\ . & . & . & . & . & . & . \end{pmatrix}$$

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Definition of the combined TRF

- The normal equation system described in the previous slides is singular and has a rank diffciency of 7 parameters.
- The 7 parameters are the defining parameters of the combind TRF *c*: origin (3 components), scale (1 component) and orientation (3 components).
- The combined TRF *c*, could be defined by, e.g.:
  - Fixing to given values 7 parameters among those to be estimated
  - Using minimum constraint equation over a selected set of stations of a reference TRF solution X_R.

Combination of long-term TRF solutions

The basic combination model is extended to include station velocities and is written as:

$$\begin{cases} X_s^i = X_c^i + T_s + D_s X_c^i + R_s X_c^i \\ \dot{X}_s^i = \dot{X}_c^i + \dot{T}_s + \dot{D}_s X_c^i + \dot{R}_s X_c^i \end{cases}$$

where the dotted parameters are their time derivatives.

Inputs: X_s^i, position of point i, at epoch t_s and velocities, $\dot{X}_s^i$, of individual solution s.

Outputs: combined positions X_c^i, at epoch t_s, velocities and transformation parameters T_s, D_s, R_s, at epoch t_s, from TRF s to TRF c.

In the same way as for daily or weekly TRF combination, observation and normal equations could easily be derived.

Note: this combination model is only valid at a give epoch, both for the input and output station coordinates

Stacking/cumulating TRF time series

The basic combination model is written as:

$$\boxed{X_s^i = X_c^i(t_0) + (t_s - t_0)\dot{X} + T_s + D_s X_c^i + R_s X_c^i}$$

Inputs: Time series of station positions, X_s^i, at different epochs t_s.
Outputs: combined positions X_c^i at epoch t_0, velocities and transformation parameters T_s, D_s, R_s from TRF s to TRF c.

Here also, observation and normal equations are construted and solved by least squares adjustment.

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